Dean-Kawasaki Dynamics: Ill-Posedness vs. Triviality

Max von Renesse (Leipzig)

Joint works with

Vitalii Konarovskyi , Tobias Lehmann

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Dean-Kawasaki Equation

$$d\mu_t = \beta \Delta \mu_t dt + \mathrm{div}(\sqrt{\mu_t} d\vec{W}_t),$$

$$\mu_0, \mu_t \in \mathcal{P}(\mathbb{R}^d)$$

$$d\vec{W}_t = \text{vector valued space-time white noise}$$

Model for super-cooled liquids and glassy materials in physics[1].
Prototype model in the theory of "fluctuating hydrodynamics"[2].

[1] [Dean 96,Kawasaki 73]
[2] e.g. [Spohn 91, Bertini et. al. 2015]

Definition

μ_t is a weak solution to DK iff $\mu_t \in \mathcal{P}(\mathbb{R}^d)$ and for all $\in C_c^\infty(\mathbb{R}^d)$

$$M_t^f := \langle \mu_t, f \rangle - \beta \int_O^t \langle \mu_s, \Delta f \rangle \, ds$$

is a martingale with quadratic variation process

$$[M^f, M^f]_t = \frac{1}{2} \int_0^t \langle \mu_s, |\nabla f|^2 \rangle ds.$$

Remark

Similar Structure to 'Super Brownian Motion'/'Dawson-Watanabe Process'

$$d\mu = \beta \Delta \mu dt + \sqrt{\mu} dW_t$$

Ill-Posedness vs. Triviality of DK equation

Theorem ([Lehmann/Konarovskyi/vR '19][3])

The Dean-Kawasaki equation

$$d\mu_t = \beta\,\Delta\mu_t dt + \text{div}(\sqrt{\mu_t} d\vec{W}_t),$$

admits weak a.s. pathwise continuous solutions in $\mathcal{P}(\mathbb{R}^d)$ *iff* $\beta = 2n$ *for some* $n \in \mathbb{N}$, *in which case it is given by*

$$\mu_t = \frac{1}{n}\sum_{j=1}^{n} \delta_{B_t^{(j)}}$$

with independent $\{B^{(j)}_\cdot\}_{j=1}^n$ *BMs in* $\mathbb{R}^d$.

[3] *Electr. Comm. Probab. ('19)*

Ingredients of Proof

Lemma (Log-Laplace-Duality)

Assume μ_t solves DK SPDE in weak sense then for $f \in L_0(\mathbb{R}^d)$

$$\log \mathbb{E}(e^{-\langle \mu_t, f \rangle}) = -\langle \mu_0, V_t f \rangle$$

where

$$V_t f(x) = -2\beta \ln P_{\beta t} e^{-\frac{1}{2\beta} f}(x).$$

for $P_t = e^{\frac{1}{2}\Delta}$ (heat semigroup)[4].

[4] $f_t = V_t f$ solves $\partial_t f_t = \beta \Delta f_t - \frac{1}{2} |\nabla f_t|^2$.

Ingredients of Proof (cont'd)

Lemma

Let X *be a non negative random variable, such that for each* $n \in \mathbb{N}^0$

$$g(s) = \mathbb{E}s^X = \sum_{k=0}^{n} s^k p_k + o(s^n), \quad \textit{as} \;\; s \to 0+,$$

for some sequence $\{p_k, \;\; k \in \mathbb{N}_0\}$*. Then* $X \in \mathbb{N}_0$ *a.s.,* $p_k \geq 0$ *and*

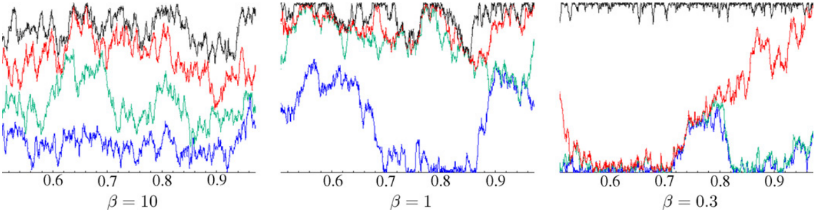
$$\mathbb{P}\{X = k\} = p_k \quad \forall k \in \mathbb{N}.$$

Anharmonic Chain Model with Electrostatic Attraction

$$dx_t^i = (\frac{\beta}{N} - 1)\left(\frac{1}{x_t^i - x_t^{i-1}} - \frac{1}{x_t^{i+1} - x_t^i}\right)dt + \sqrt{2}dw_t^i + dl_t^{i-1} - dl_t^i,$$

Local times

$$dl_t^i \geq, \quad l_t^i = \int_0^t \chi_{\{x_s^i = x_s^{i+1}\}} dl_s^i.$$



Theorem (Andres/v.R., *JFA*, '10)

For

$$\mu_t^N = \frac{1}{N-1} \sum_{i=1}^{N-1} \delta_{x^i_{N\cdot t}} \in \mathcal{P}(\mathbb{R}),$$

then

$$(\mu_\cdot^N) \Longrightarrow (\mu_\cdot) \text{ in } C_{\mathbb{R}_+}\big(\mathcal{P}(\mathbb{R}), \tau_w\big)$$

where

$$d\mu = \beta \Delta \mu dt + \Gamma(\mu) dt + \text{div}(\sqrt{\mu} dW),$$

with

$$\langle f, \Gamma(\mu) \rangle = \sum_{I \in \text{gaps}(\mu)} \left[\frac{f''(I_+) + f''(I-)}{2} - \frac{f'(I_+) - f'(I_+)}{|I|} \right]$$

Remark on Invariant Measure [v.R/Sturm AOP 2009]

Invariant measure on $\mathcal{P}([0,1])$

$$\mathfrak{P}^\beta(d\mu) = \frac{1}{Z} e^{-\beta \, \mathsf{Ent}(\mu)} \, \text{'vol'}^{\mathcal{P}}(d\mu),$$

i.e.

$$\langle f, \mu \rangle := \int_0^1 f(D_t^\beta) dt$$

where $D_t^\beta = \frac{\gamma_{\beta \cdot t}}{\gamma^\beta}$ Dirichlet process with Parameter β.

Short time asymptotics/Varadhan Formula [v.R/Sturm AOP 2009]

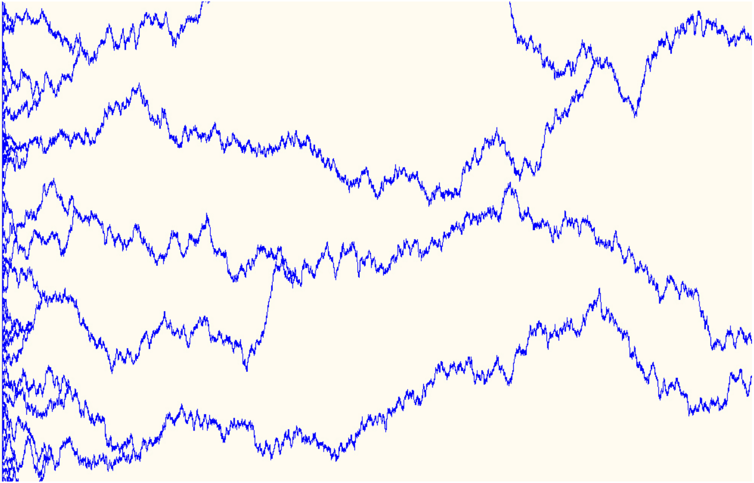
Theorem

For rel. open $A, B \subset \mathcal{P}_2^\eta$ *with* $0 < \Xi^\eta(A)\Xi^\eta(B) < \infty$ *it holds that*

$$\lim_{t \to 0} t \cdot \ln \mathbb{P}(\mu_0 \in A, \mu_t \in B) = -\frac{d_{\mathcal{W}}^2(A, B)}{2},$$

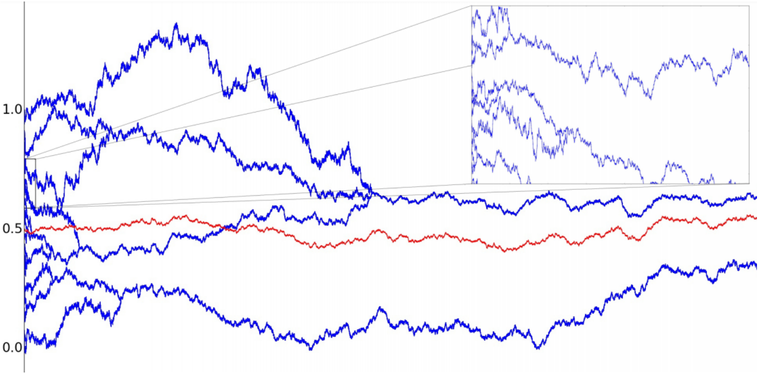
where $d_{\mathcal{W}}(A, B) = \inf_{(\rho,\lambda) \in A \times B} d_{\mathcal{W}}(\rho, \lambda)$.

Arratia Flow[5]



[5] Richard A. Arratia, PhD Thesis U Madison/Wisc., (1979)

Case $\beta = 0$: Modified Arratia Flow[6]



[6] [Konarovskyi, AOP +17], [Konarovskyi/vR, CPAM 18]

Theorem ([Konarovskyi, AOP 17+])

There is a process $y = (y(u,t)|u \in [0,1], t \geq 0) \in D([0,1], C([0,T])$

(C1) *for all* $u \in [0,1]$, *the process* $y(u,\cdot)$ *is a continuous square integrable martingale with respect to the filtration*

$$\mathcal{F}_t = \sigma(y(u,s),\ u \in [0,1],\ s \leq t), \quad t \in [0,T];$$

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(C4) *for all* $u, v \in [0,1]$,

$$[y(u,\cdot), y(v,\cdot)]_t = \int_0^t \frac{\mathbb{I}_{\{\tau_{u,v} \leq s\}} ds}{m(u,s)},$$

where $m(u,t) = |\{v : \ y(v,t) = y(u,t)\}|$,
$\tau_{u,v} = \inf\{t : \ y(u,t) = y(v,t)\} \wedge T$.

Induced Random Flow on $\mathcal{P}(\mathbb{R})$

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Theorem ([Konarovskyi/vR, CPAM '18])

Let $\mu_t := y(.,t)_\# \lambda_{\llcorner [0,1]}$, *then*

$$d\mu_t = \frac{1}{2}\Delta\mu_t^* dt + \mathrm{div}\,\left(\sqrt{\mu_t} dW_t\right)$$

where

$$\langle f, \Delta\mu_t^*\rangle = \sum_{x \in supp(\mu_t)} f''(x).$$

$\rightsquigarrow$ *Solution to (corrected) DK-equation for* $\beta = 0$. *Non reversible!*

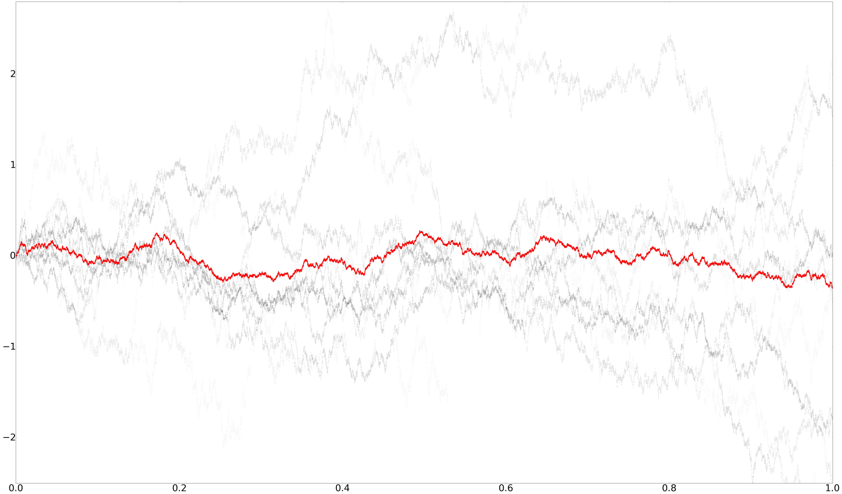
Varadhan Formula for Modified Arratia Flow

Theorem ([op. cit.])

For 'nice' measurable $A \subset \mathcal{P}(\mathbb{R})$

$$\lim_{\epsilon \to 0} \epsilon \log P(\mu_\epsilon \in A) = -\frac{d_{\mathcal{W}}^2(\lambda_{\llcorner [0,1]}, A)}{2}.$$

Case $\beta = 0$ Modified Arratia Flow with Splitting[7]



[7] [Konarovskyi,vR '17], arxiv 1709.02839

Construction via Dirichlet Forms on $\mathcal{P}(\mathbb{R})$

Parameterization of $\mathcal{P}(\mathbb{R})$ by inverse distribution functions

$$L_2^{\uparrow} := \{g \in L^2(0,1) \,|\, g \text{ non decreasing } \}$$

$$\mathcal{P}_2(\mathbb{R}) \ni \quad \mu \leftrightsquigarrow g_\mu := \mu((-\infty,\cdot])^{-1} \quad \in L_2^{\uparrow}$$

Isometry

$$\|g_\mu - g_\nu\|_{L^2} = d_{\mathcal{W}}(\mu,\nu) \quad \text{(Wasserstein-Distance)}$$

The boundary of $L_2^\uparrow \subset L^2[0,1]$ as simplicial complex

'Pure' boundary of $L_2^\uparrow$: Collection of step functions

$$\chi \in \partial L_2^\uparrow$$

$$\chi_n(q,x) = \sum_{i=1}^{n} x_i \mathbb{I}_{[q_{i-1},q_i)}, \quad x \in E^n, \quad q \in Q^n,$$

- $\chi_n(q,x)$: Element of n-dimensional face of $\partial L_2^\uparrow$, common boundary point of uncountably many faces of dimension $n+1$.
- $\partial L_2^\uparrow \subset L^2$ dense.

The Invariant Measure

Parametrization of step functions $\chi \in L_2^{\uparrow}$:

$$\chi_n(q,x) = \sum_{i=1}^{n} x_i \mathbb{I}_{[q_{i-1},q_i)} + x_n \mathbb{I}_{\{1\}}, \quad x \in E^n, \quad q \in Q^n,$$

Uniform measure w.r.t. x-coordinate

$$\nu_n(q,A) = \lambda_n\{x : \ \chi_n(q,x) \in A\}, \quad A \in \mathcal{B}(L_2^{\uparrow}).$$

Choose $\xi \in L_2^{\uparrow}$ (model parameter defining splitting rates) and set

$$\Xi_n(A) = \int_{Q^n} \left(\prod_{i=1}^{n} (q_i - q_{i-1}) \right) \nu_n(q,A) d\xi^{\otimes(n-1)}(q), \quad A \in \mathcal{B}(L_2^{\uparrow}),$$

$$\boxed{\Xi := \sum_{n=1}^{\infty} \Xi_n} \quad \text{measure on } L_2^{\uparrow}.$$

Gradient Operator and Pre-Dirichlet Form

Natural 'tangential' gradient on each of the faces of the simplex $L_2^{\uparrow}$

$$D\,U(g) := \mathrm{pr}^{L^2}_{\sigma(g)}\left[\nabla^{L_2} U_{|g}\right].$$

Define (pre-)Dirichlet form

$$\mathcal{E}(U,U) = \int_{L_2^{\uparrow}} \|\,D\,U(g)\|^2_{L^2}\,\Xi(dg)$$

Integration by Parts Formula

Theorem

Let $U, V \in \mathcal{FC}$. *Then*

$$\int_{L_2^{\uparrow}} \langle \mathcal{D}U(g), \mathcal{D}V(g)\rangle \Xi(dg) = -\int_{L_2^{\uparrow}} L_0 U(g) V(g) \Xi(dg) - \int_{L_2^{\uparrow}} V(g) \langle \nabla^{L_2} U(g) - \mathcal{D}U(g), \xi \rangle \Xi(dg).$$

where, for $g = \chi_n(q, x)$,

$$L_0 U(g) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} U(\chi_n(q, x)) \frac{1}{(q_i - q_{i-1})}$$

Main result

Theorem (Konarovskyi/v.R. '17)

For $\eta \in \mathcal{P}(\mathbb{R})$ there exists a measure Ξ^η on $\mathcal{P}^2(\mathbb{R})$ with $spt(\Xi^\eta) = \mathcal{P}_2^\eta(\mathbb{R})$ such that the quadratic form

$$\mathcal{E}(F,F) = \int_{\mathcal{P}_2^\eta(\mathbb{R})} \|D_{|\mu} F(.)\|^2_{L^2([0,1])}\, \Xi^\eta(d\mu), \quad F \in \mathcal{F}$$

is closable on $L^2(\mathcal{P}_2^\eta, \Xi^\eta)$, its closure being a strongly local quasi-regular Dirchlet form on $L^2(\mathcal{P}_2^\eta, \Xi^\eta)$. - Let $(\mu_t)_{t\in[0,\zeta[}$ the properly associated Ξ^η-symmetric diffusion process with life time $\zeta > 0$, then

i) *For all $t \in [0,\zeta[$ it holds that $\mu_t \in \mathcal{P}_2^a$ almost surely.*

ii) *For all $f \in C_0^\infty(\mathbb{R})$ the process*
$$M^f := \langle \mu_t, f\rangle - \int_0^t \langle \mu_s^*, f''\rangle ds, \quad \mu^* = \sum_{x\in \mathrm{spt}\mu} \delta_x$$
is a local martingale with finite quadratic variation process
$$[M^f]_t = \int_0^t \langle \mu_s, (f')^2\rangle ds.$$

Theorem (cont'd)

iii) *For all* $h \in C^\infty([0,1])$ *the process*

$$\tilde{M}^h := \langle g_\mu, h\rangle + \int_0^t \langle \mathrm{pr}^\perp_{g_{\mu_s}} h, g_\xi\rangle ds$$

is a local martingale with finite quadratic variation process

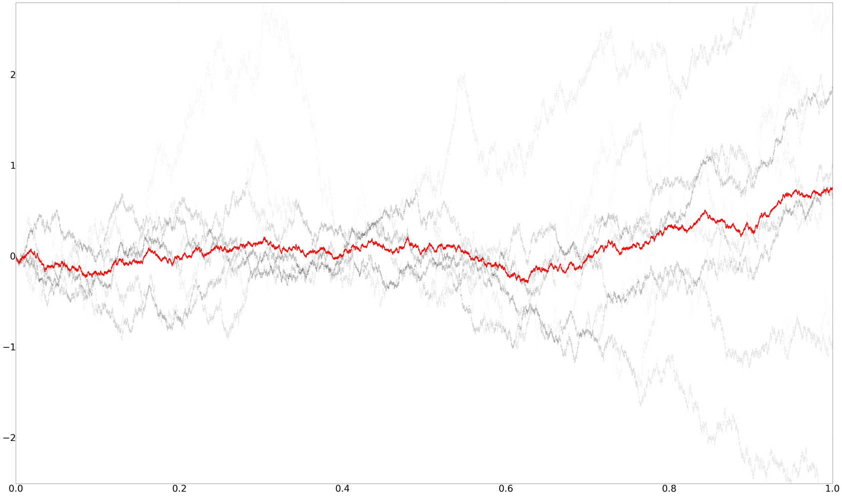
$$[\tilde{M}^h]_t = \int_0^t \|\mathrm{pr}_{g_{\mu_s}} h\|^2_{L^2[0,1]} ds.$$

iv) *For rel. open* $A, B \subset \mathcal{P}_2^\eta$ *with* $0 < \Xi^\eta(A)\Xi^\eta(B) < \infty$ *it holds that*

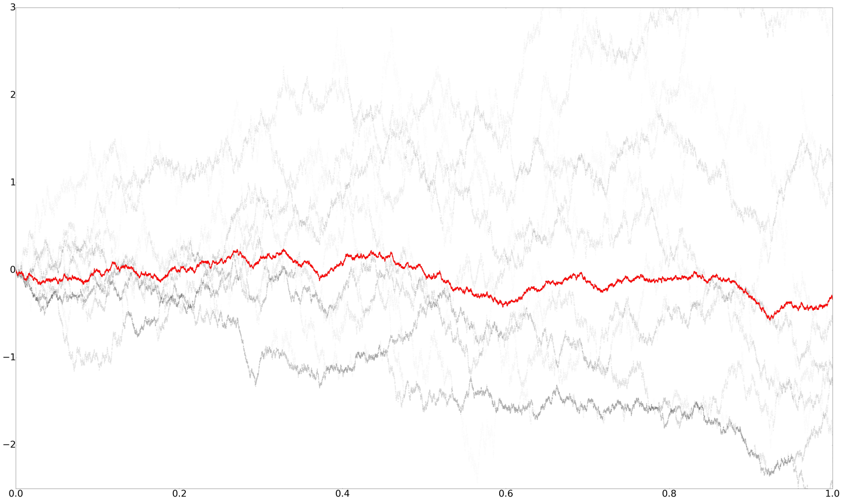
$$\lim_{t\to 0} t \cdot \ln \mathbb{P}(\mu_0 \in A, \mu_t \in B) = -\frac{d^2_{\mathcal{W}}(A,B)}{2},$$

where $d_{\mathcal{W}}(A,B) = \inf_{(\rho,\lambda)\in A\times B} d_{\mathcal{W}}(\rho,\lambda)$.

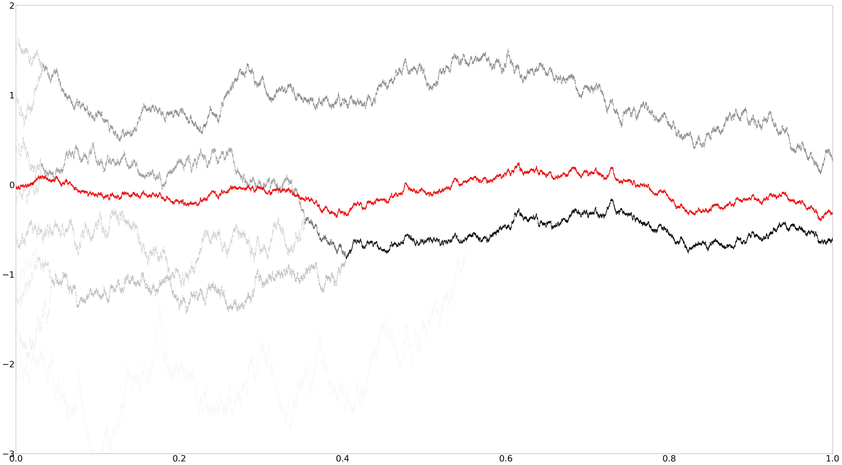
Small Fragmentation Parameter



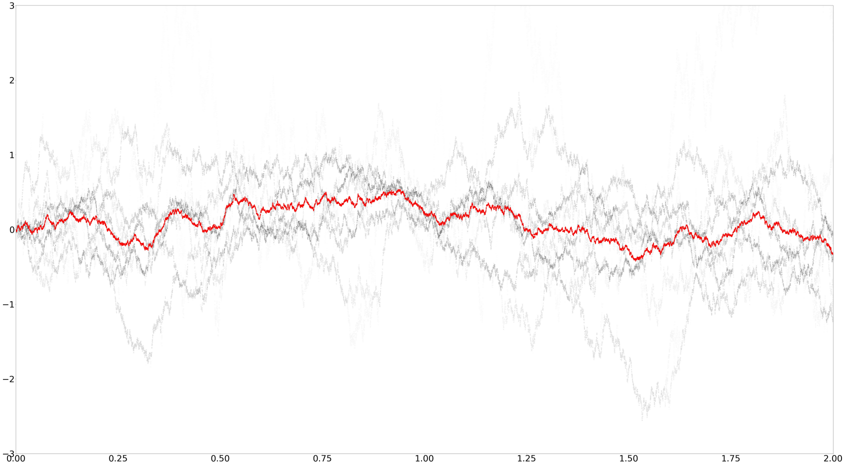
Inhomogeneous Fragmentation Dynamics



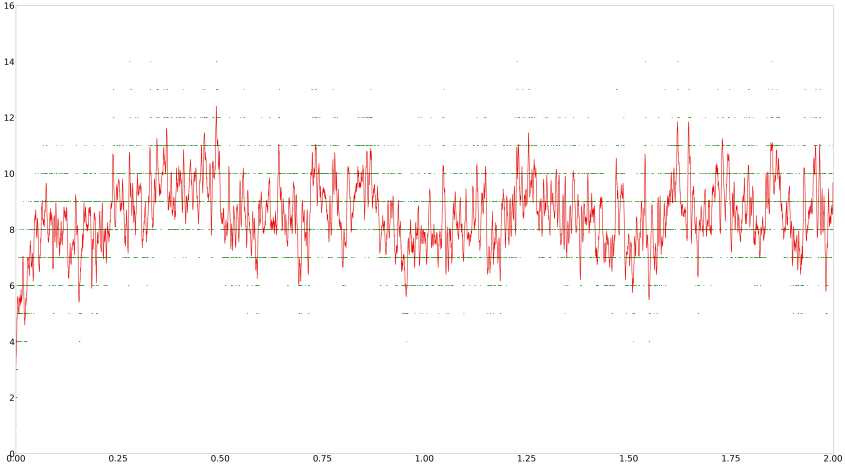
Pure coalescence/no fragmentation



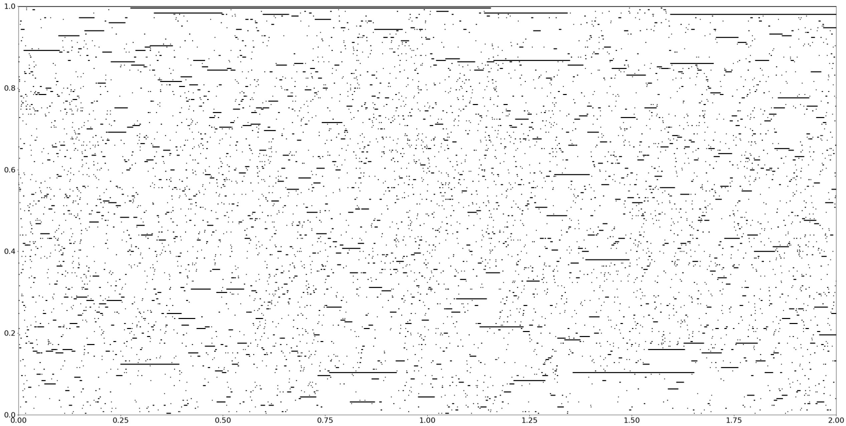
Ornstein-Uhlenbeck Version



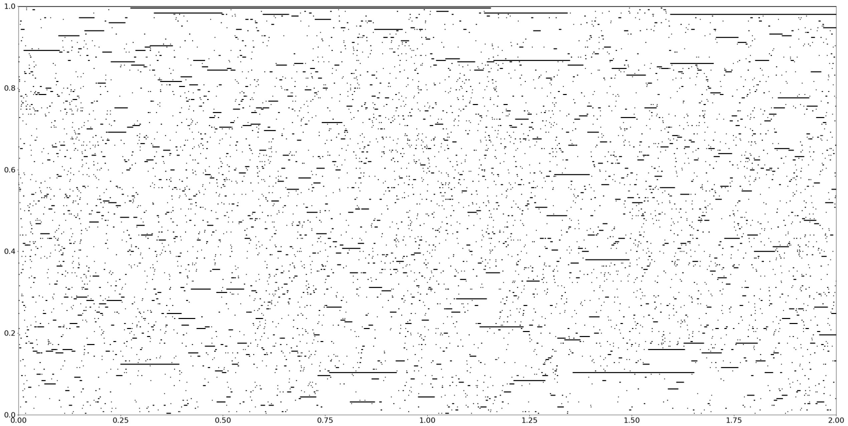
Ornstein-Uhlenbeck Version - Cluster Number Dynamics



Ornstein-Uhlenbeck Version - Partition Dynamics



Ornstein-Uhlenbeck Version - Partition Dynamics



Thank you! □