(Nonlinear) Fluctuating Hydrodynamics and Physics on Mesoscopic Scales

joint work with Herbert Spohn
Part 2

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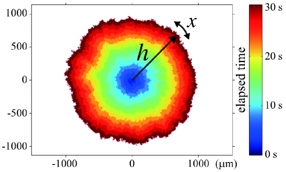
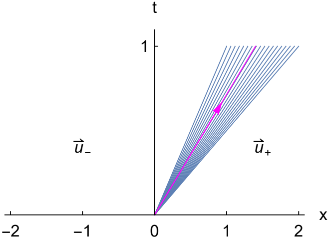


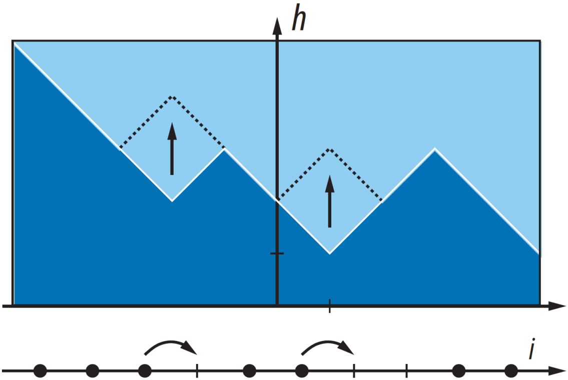
First Berlin – Leipzig Workshop on Fluctuating Hydrodynamics

Tracy-Widom distribution in non-equilibrium processes



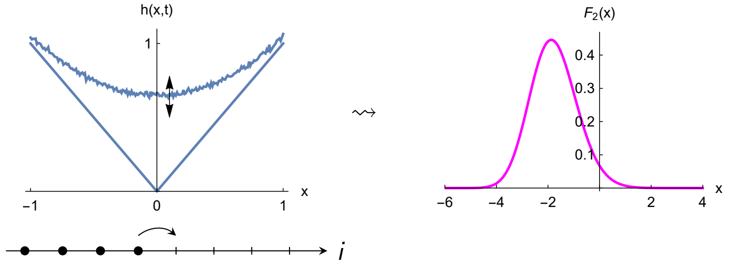
Discrete model: TASEP particle process

Totally Asymmetric Simple Exclusion Process:



height function $h(x,t) \leftrightarrow$ time-integrated current $\Phi(x,t)$

Tracy-Widom distribution for wedge initial geometry



(T)ASEP particle process: integrated current $\Phi(x,t)$ converges to GUE Tracy-Widom distribution $F_2(x)$:

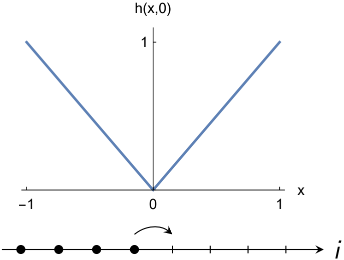
$$\Phi(0,t) \simeq \frac{1}{4}t - \left(2^{-4}t\right)^{1/3} \xi_{\mathrm{TW}}$$

(Compare with GUE ensemble $\lambda_N \simeq 2N + N^{1/3}\xi_{\mathrm{TW}}$ for largest eigenvalue of complex hermitian matrices $A \sim Z^{-1}\exp[-\frac{1}{2N}\mathrm{tr}A^2]$)

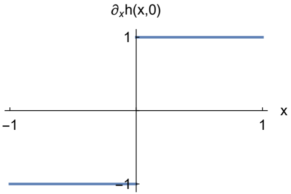
Johansson 2000 for TASEP, Tracy and Widom 2009 for ASEP

Wedge initial geometry translated to conserved fields

Derivative $u(x,t) = \partial_x h(x,t)$ has a jump for "wedge" initial conditions:



$\rightsquigarrow$



$\rightsquigarrow$ "domain wall" initial conditions, i.e., solve *Riemann problem*

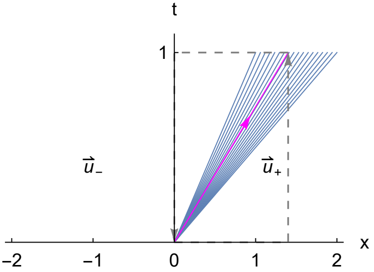
$$\vec{u}(x,0) = \vec{u}_- \quad \text{for} \quad x \leq 0, \qquad \vec{u}(x,0) = \vec{u}_+ \quad \text{for} \quad x > 0$$

Integrated current and TW for anharmonic chains

Integrated current for anharmonic chains (corresponds to $h(x, t)$):

$$\Phi(x, t) = \int_0^t \mathrm{j}(x, t')\,\mathrm{d}t' - \int_0^x u(x', 0)\,\mathrm{d}x'$$

(path independent since $(-u, \mathrm{j})$ is curl free: $\partial_t u + \partial_x \mathrm{j} = 0$)



In analogy to (T)ASEP, for integration within rarefaction wave: expecting

$$\Phi(ct, t) \simeq a_0 t + (\Gamma t)^{1/3} \xi_{\mathrm{TW}}$$

Integrated current: Transformation to normal modes

Projected current components (with asymptotic value subtracted):

$$\Phi^{\sharp}_{\sigma}(t) = \left\langle \tilde{\psi}_{\sigma} | \vec{\Phi}(ct, t) - t(\vec{\mathrm{j}}(\vec{u}_c) - c\vec{u}_c) \right\rangle$$

with $\tilde{\psi}_{\sigma}$ the left eigenvectors of $A(\vec{u}_c)$

$\Phi^{\sharp}_{1}(t)$ results from perturbations propagating with velocity c along the magenta observation ray $\rightsquigarrow$ expecting

$$\Phi^{\sharp}_{1}(t) \simeq (\Gamma_1 t)^{1/3} \xi_{\mathrm{TW}}$$

Integrated current: Transformation to normal modes

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$$\Phi^\sharp_1(t) \simeq (\Gamma_1 t)^{1/3} \xi_{\mathrm{TW}}$$

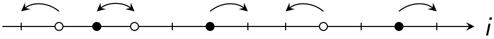
Components $\sigma \neq 1$ pick up samples from essentially independent space-time regions $\rightsquigarrow$ central limit type behavior

$$\Phi^\sharp_\sigma(t) \simeq (\Gamma_\sigma t)^{1/2} \xi_{\mathrm{G}}, \qquad \sigma \neq 1,$$

with ξ_{G} a normalized Gaussian random variable

Application to the LeRoux lattice gas
(Fritz and Tóth 2004)

$+1$ and -1 particles $\rightsquigarrow$ occupation $\eta_i \in \{-1, 0, 1\}$ (0: empty)



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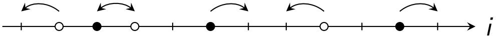

Steady state parameters $\vec{u} = (\rho, v)$ with ρ: hole density, v: velocity

$$\mathbb{P}_{\rho,v}(\eta_i = 0) = \rho, \qquad \mathbb{P}_{\rho,v}(\eta_i = \pm 1) = \frac{1}{2}(1 - \rho \pm v)$$

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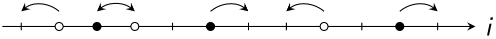

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Conservation law:

$$\partial_t \vec{u} + \partial_x \vec{\mathrm{j}}(\vec{u}) = 0 \quad \text{with} \quad \vec{\mathrm{j}}(\vec{u}) = -(\rho\, v, \rho + v^2 - 1)$$

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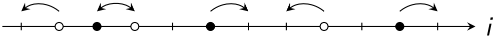

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$$\text{Jacobian } A = \frac{\partial \vec{\mathrm{j}}(\vec{u})}{\partial \vec{u}} = -\begin{pmatrix} v & \rho \\ 1 & 2v \end{pmatrix}$$

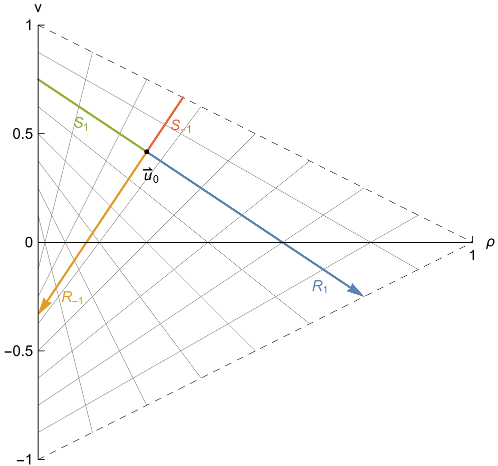
Diagonalization: $A\psi_\sigma = c_\sigma \psi_\sigma$ $(\sigma = \pm 1)$ with

$$c_\sigma = -\tfrac{3}{2}v + \tfrac{1}{2}\sigma\sqrt{4\rho + v^2}, \quad \psi_\sigma = Z_\sigma^{-1}\begin{pmatrix} 2\sigma\rho \\ \sigma v - \sqrt{4\rho + v^2} \end{pmatrix}$$

Riemann problem for LeRoux

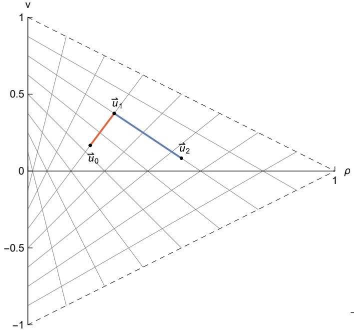
Rarefaction curves: solve Cauchy problem $\partial_\tau \vec{u} = \psi_\sigma(\vec{u})$

Shock curves: Rankine-Hugoniot jump condition $\lambda(\vec{u} - \vec{u}_0) = \vec{\mathrm{j}}(\vec{u}) - \vec{\mathrm{j}}(\vec{u}_0)$

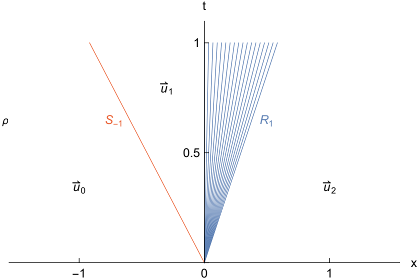


Mendl and Spohn 2016, 2017, cf. Bressan 2013

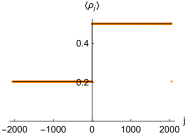
Riemann problem for LeRoux



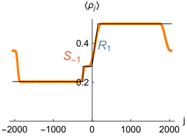
(a) path in state space



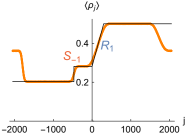
(b) shock and rarefaction wave



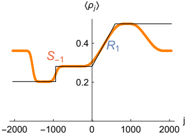
(c) density, $t = 0$



(d) $t = 256$



(e) $t = 512$



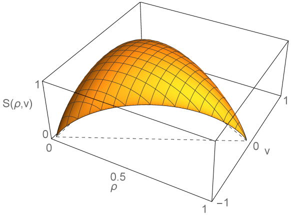
(f) $t = 1024$

Mendl and Spohn 2016, 2017

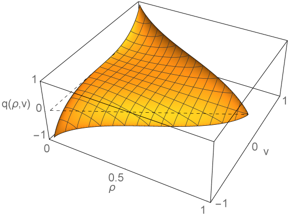
Entropy

Entropy: $$S(\rho, v) = -\sum_{\eta=-1}^{1} \mathbb{P}_{\rho,v}(\eta) \log \mathbb{P}_{\rho,v}(\eta)$$

Entropy flux: $$q(\rho, v) = v + \sum_{\eta=-1}^{1} (v-\eta)\mathbb{P}_{\rho,v}(\eta) \log \mathbb{P}_{\rho,v}(\eta)$$



(a) entropy $S(q, v)$



(b) entropy flux $q(\rho, v)$

Entropy

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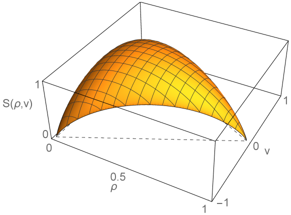

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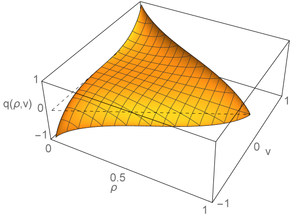


(b) entropy flux $q(\rho, v)$

Entropy inequality admissibility condition (in the sense of distributions):

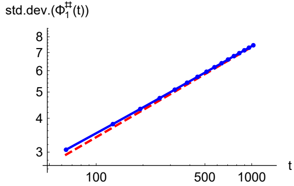
$$\partial_t S(\vec{u}) + \partial_x q(\vec{u}) = \Delta S(\vec{u}) \geq 0$$

Can derive $\Delta S(\vec{u}) = 0$ for rarefaction, $\Delta S(\vec{u}) \geq 0$ for shock

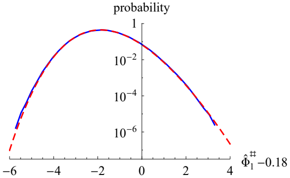
Mendl and Spohn 2016, 2017, cf. Bressan 2013

Simulation results for LeRoux: integrated currents

Statistical distribution of $(\Gamma_1 t)^{-1/3}\, \Phi^\sharp_1(t) \rightsquigarrow$ Tracy-Widom



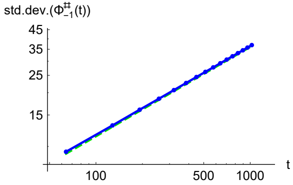
(a) red: $\sigma_{\mathrm{GUE}}(\Gamma_1 t)^{1/3}$



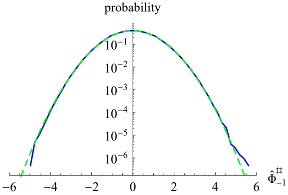
(b) $(\Gamma_1 t)^{-1/3}\, \Phi^\sharp_1(t)$, TW

$\Gamma_1 = |G^1_{11}|$ (entry of coupling matrix)

Projection $(\Gamma_{-1} t)^{-1/2}\, \Phi^\sharp_{-1}(t) \rightsquigarrow$ Gauss



(a) green: $\sim t^{1/2}$



(b) $(1.34t)^{-1/2}\, \Phi^\sharp_{-1}(t)$, Gauss

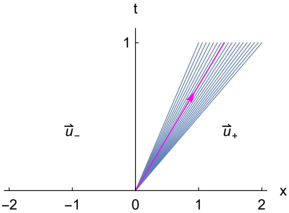
Application to hard-point chain: Riemann problem

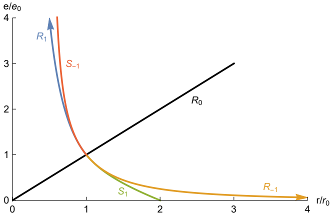
Rarefaction curves: solve Cauchy problem

$$\partial_\tau \vec{u} = \psi_\alpha(\vec{u}), \quad \alpha = 0, \pm 1$$

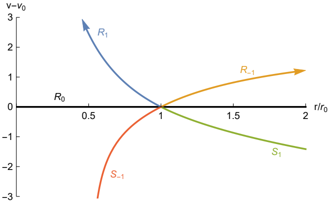
with ψ_α the right eigenvectors of $A = \frac{\partial \mathrm{j}(\vec{u})}{\partial \vec{u}}$

$$\partial_\tau \begin{pmatrix} r \\ v \\ \mathfrak{e} \end{pmatrix} = \begin{pmatrix} -\frac{1}{m}\sigma \\ c \\ \frac{1}{m}\sigma P + \frac{\nu}{m} c \end{pmatrix}, \quad P : \text{pressure}$$





(a) energy vs. stretch

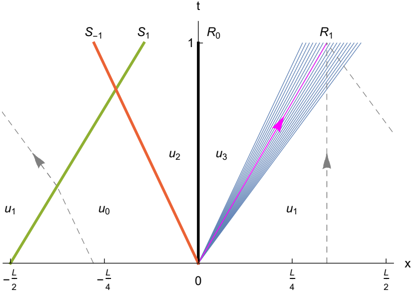


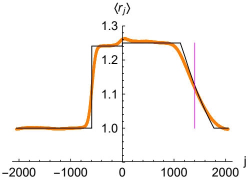
(b) velocity vs. stretch

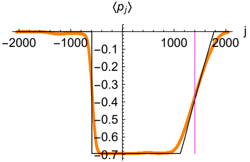
Mendl and Spohn 2016, 2017, cf. Bressan 2013

Riemann problem solution for periodic boundaries

Analytical solution of the *periodic* Riemann problem:







orange: molecular dynamics

Mendl and Spohn 2016, 2017, cf. Bressan 2013

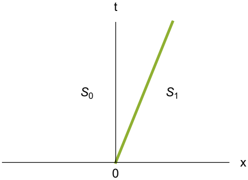
Entropy

Entropy:

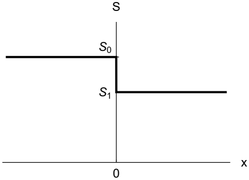
$$S(r,e) = \beta(rP + e) + \frac{1}{2}\log(2\pi m) - \frac{1}{2}\log\beta + \log Z(P,\beta)$$

with $P = P(r,e)$, $\beta = \beta(r,e)$

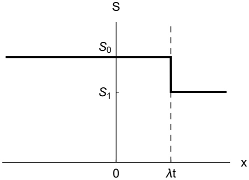
Entropy flux vanishes for anharmonic chains



(a) shock x-t profile



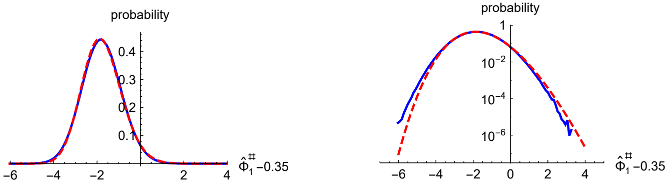
(b) entropy at $t = 0$



(c) entropy at $t > 0$

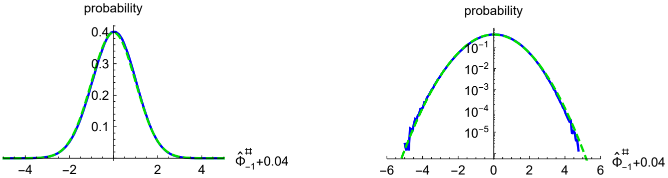
Simulation of hard-point chain: integrated currents

Statistical distribution of $(\Gamma_1 t)^{-1/3}\,\Phi_1^\sharp(t) \rightsquigarrow$ Tracy-Widom



(a) linear scale (b) log scale

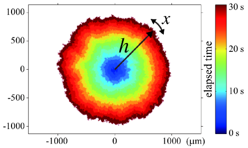
Projection $(\Gamma_{-1} t)^{-1/2}\,\Phi_{-1}^\sharp(t) \rightsquigarrow$ Gauss



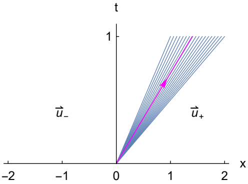
(a) linear scale (b) log scale

Summary and conclusions

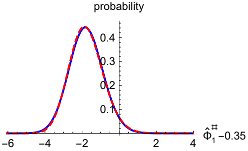
- Nonlinear fluctuating hydrodynamics $\leftrightarrow$ KPZ framework



- Tracy-Widom GUE distribution for "domain wall" initial condition



- Observing Tracy-Widom GUE for projected current component integrated within rarefaction wave



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