# $h^*$-polynomials of dilated lattice polytopes

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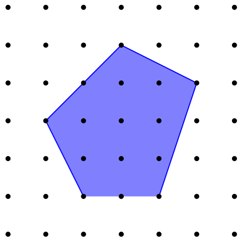
Einstein Workshop Discrete Geometry and Topology, March 13, 2018

# Lattice polytopes

A set $P \subset \mathbb{R}^d$ is a **lattice polytope** if there are $x_1, \ldots, x_m \in \mathbb{Z}^d$ with

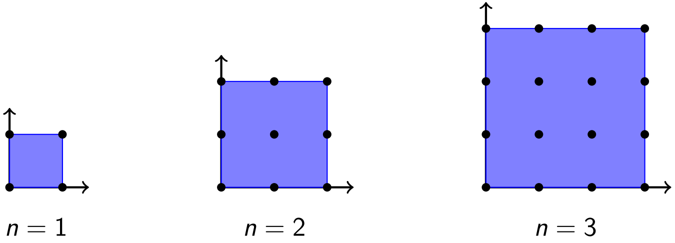
$$P \;=\; \text{conv}\{x_1, \ldots, x_m\}.$$



# Ehrhart theory

The **lattice point enumerator** or **discrete volume** of $P$ is

$$\mathrm{E}(P) \;:=\; \left|P \cap \mathbb{Z}^d\right|.$$




$$\mathrm{E}(nP) = (n+1)^2.$$

# Ehrhart theory

## Theorem (Ehrhart'62)

*For every lattice polytope $P$ in $\mathbb{R}^d$*

$$\mathrm{E}_P(n) := |nP \cap \mathbb{Z}^d|$$

*agrees with a polynomial of degree* $\dim P$ *for* $n \geq 1$.

$\mathrm{E}_P(n)$ is called the **Ehrhart polynomial** of $P$.

Various *combinatorial applications*, i.e.

- posets (order preserving maps),
- graph colorings,...

## Central Questions

- Which polynomials are Ehrhart polynomials?
- Interpretation of coefficients
- roots, ...

# Ehrhart series and $h^*$-polynomial

## Ehrhart series

The **Ehrhart series** of an $d$-dimensional lattice polytope $P \subset \mathbb{R}^d$ is defined by

$$\sum_{n \geq 0} \mathrm{E}_P(n) t^n \quad = \quad \frac{h_0^* + h_1^* t + \cdots + h_d^* t^d}{(1-t)^{d+1}} \, .$$

The numerator polynomial $h_P^*(t)$ is the $h^*$-**polynomial** of $P$. The vector $h^*(P) := (h_0^*, \ldots, h_d^*)$ is the $h^*$-**vector**.

# Ehrhart series and $h^*$-polynomial

## Ehrhart series

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## $h^*$-vector and coefficients of $\mathrm{E}_P(n)$

Expansion into a binomial basis:

$$\mathrm{E}_P(n) = h_0^* \binom{n+r}{r} + h_1^* \binom{n+r-1}{r} + \cdots + h_d^* \binom{n}{r} \, .$$

# Inequalities for the $h^*$-vector

## Theorem (Stanley '80)

*For every lattice polytope $P$ in $\mathbb{R}^d$ with $h_P^* = h_0^* + h_1^* t + \cdots + h_d^* t^d$*

$$h_i^* \geq 0$$

*for all $0 \leq i \leq d$.*

**Question**: Are there stronger inequalities for certain classes of polytopes?

Such as...

- **...Unimodality**:

$$h_0^* \leq h_1^* \leq \cdots \leq h_k^* \geq \cdots \geq h_d^* \text{ for some } k$$

- **...Log-concavity**:

$$(h_k^*)^2 \geq h_{k-1}^* h_{k+1}^* \text{ for all } k$$

- **...Real-rootedness**:

$$h_P^* = h_0^* + h_1^* t + \cdots + h_d^* t^d \quad \text{has only real roots}$$

# IDP polytopes

## Conjecture (Stanley '89)

*Every IDP polytope has a unimodal $h^*$-vector.*

A lattice polytope $P \subset \mathbb{R}^d$ has the **integer decomposition property (IDP)** if for all integers $n \geq 1$ and all $p \in nP \cap \mathbb{Z}^d$

$$p = p_1 + \cdots + p_n$$

for some $p_1, \ldots, p_n \in P \cap \mathbb{Z}^d$.

## Examples

- unimodular simplex
- lattice parallelepiped
- lattice zonotope
- $rP$ whenever $r \geq \dim P - 1$
  (Bruns, Gubeladze, Trung '97)

# Dilated lattice polytopes

### Theorem (Brenti, Welker '09; Diaconis, Fulman '09; Beck, Stapledon '10)

*Let $P$ be a $d$-dimensional lattice polytope. Then there is an $N$ such that the $h^*$-polynomial of $rP$ has only real roots for $r \geq N$.*

### Conjecture (Beck, Stapledon '10)

*Let $P$ be a $d$-dimensional lattice polytope. Then the $h^*$-polynomial of $rP$ has only real-roots whenever $r \geq d$.*

### Theorem (Higashitani '14)

*Let $P$ be a $d$-dimensional lattice polytope. Then the $h^*$-polynomial of $rP$ has log-concave coefficients whenever $r \geq \deg h^*_P$.*

### Theorem (J. '16)

*Let $P$ be a $d$-dimensional lattice polytope. Then the $h^*$-polynomial of $rP$ has only real roots whenever $r \geq \deg h^*_P$.*

# Interlacing polynomials

- Proof of Kadison-Singer-Problem from 1959 (Marcus, Spielman, Srivastava '15)
- Real-rootedness of independence polynomials of claw-free graphs (Chudnowski, Seymour '07)
  compatible polynomials, common interlacers
- Real-rootedness of $s$-Eulerian polynomials (Savage, Visontai '15)
  $h^*$-polynomial of $s$-Lecture hall polytopes are real-rooted

Further literature: Bränden '14, Fisk '08, Braun '15

# Interlacing polynomials

# Interlacing polynomials

## Definition

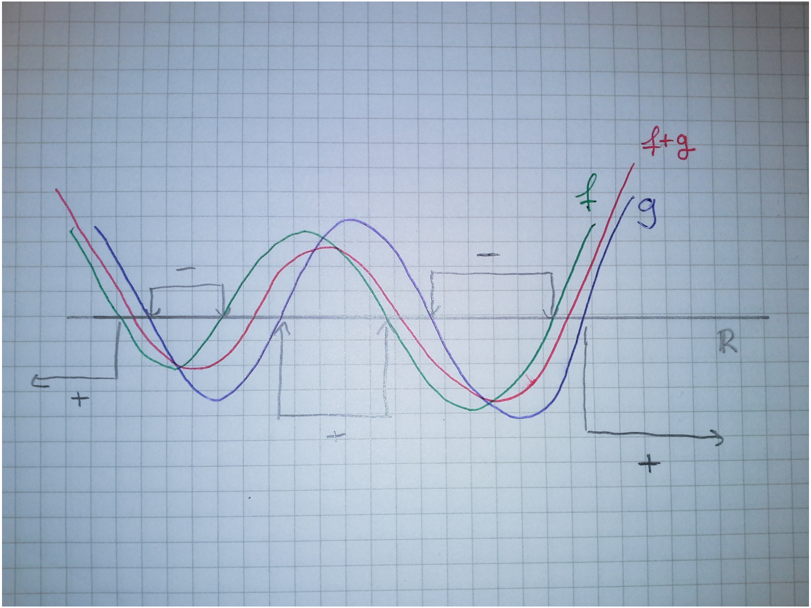
Let $a, b, t_1, \ldots, t_n, s_1, \ldots, s_m \in \mathbb{R}$. Then $f = a \prod_{i=1}^{m}(t - s_i)$ **interlaces** $g = b \prod_{i=1}^{n}(t - t_i)$ and we write $f \preceq g$ if

$$\cdots \leq s_2 \leq t_2 \leq s_1 \leq t_1$$

**Properties**

- $f \preceq g$ if and only if $cf \preceq dg$ for all $c, d \neq 0$.
- $\deg f \leq \deg g \leq \deg f + 1$
- $\alpha f + \beta g$ real-rooted for all $\alpha, \beta \in \mathbb{R}$

# Interlacing polynomials



# Polynomials with only nonpositive, real roots

## Lemma (Wagner '00)

*Let $f, g, h \in \mathbb{R}[t]$ be real-rooted polynomials with only nonpositive, real roots and positive leading coefficients. Then*

(i) *if $f \preceq h$ and $g \preceq h$ then $f + g \preceq h$.*

(ii) *if $h \preceq f$ and $h \preceq g$ then $h \preceq f + g$.*

(iii) *$g \preceq f$ if and only if $f \preceq tg$.*

# Interlacing sequences of polynomials

## Definition

A sequence $f_1, \ldots, f_m$ is called interlacing if

$$f_i \preceq f_j \qquad \text{whenever } i \leq j\,.$$

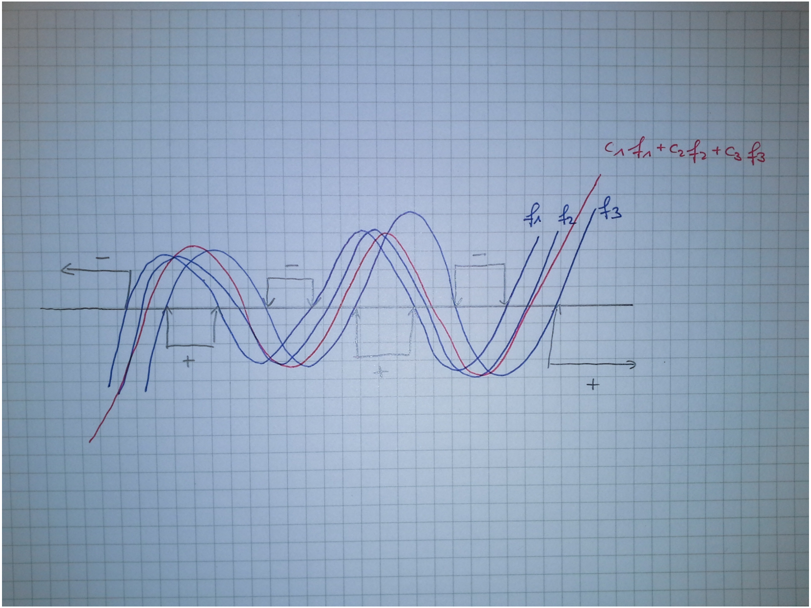
## Lemma

*Let $f_1, \ldots, f_m$ be an interlacing polynomials with only nonnegative coefficients. Then*

$$c_1 f_1 \ + \ c_2 f_2 \ + \ \cdots \ + c_m f_m$$

*is real-rooted for all $c_1, \ldots, c_m \geq 0$.*

# Interlacing sequences of polynomials



# Constructing interlacing sequences

## Proposition (Fisk '08; Savage, Visontai '15)

*Let* $f_1, \cdots, f_m$ *be a sequence of interlacing polynomials with only negative roots and positive leading coefficients. For all* $1 \leq l \leq m$ *let*

$$g_l = tf_1 + \cdots + tf_{l-1} + f_l + \cdots + f_m.$$

*Then also* $g_1, \cdots, g_m$ *are interlacing, have only negative roots and positive leading coefficients.*

# Linear operators preserving interlacing sequences

Let $\mathcal{F}_+^n$ the collection of all interlacing sequences of polynomials with only nonnegative coefficients of length $n$.
When does a matrix $G = (G_{i,j}(t)) \in \mathbb{R}[t]^{m\times n}$ map $\mathcal{F}_+^n$ to $\mathcal{F}_+^m$ by $G \cdot (f_1, \ldots, f_n)^T$?

## Theorem (Brändén '15)

*Let $G = (G_{i,j}(t)) \in \mathbb{R}[t]^{m\times n}$. Then $G \colon \mathcal{F}_+^n \to \mathcal{F}_+^m$ if and only if*

(i) $(G_{i,j}(t))$ *has nonnegative entries for all $i \in [n], j \in [m]$, and*

(ii) *For all $\lambda, \mu > 0$, $1 \leq i < j \leq n$, $1 \leq k < l \leq n$*

$$(\lambda t + \mu)G_{k,j}(t) + G_{l,j}(t) \preceq (\lambda t + \mu)G_{k,i}(t) + G_{l,i}(t)\,.$$

# Example

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ t & 1 & 1 & \cdots & 1 \\ t & t & 1 & \cdots & 1 \\ \vdots & \vdots & & & \vdots \\ t & t & \cdots & t & t \end{pmatrix} \quad \in \mathbb{R}[x]^{(n+1)\times n}$$

(i) All entries have nonnegative coefficients ✓

**Submatrices:**

$$M = \begin{matrix} & i & j \\ k & G_{k,i}(t) & G_{k,j}(t) \\ l & G_{l,i}(t) & G_{l,j}(t) \end{matrix} \quad : \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ t & 1 \end{pmatrix} \quad \begin{pmatrix} t & 1 \\ t & t \end{pmatrix} \quad \begin{pmatrix} t & t \\ t & t \end{pmatrix}$$

(ii) $(\lambda t + \mu) G_{k,j}(t) + G_{l,j}(t) \preceq (\lambda t + \mu) G_{k,i}(t) + G_{l,i}(t)$

$$(\lambda + 1)t + \mu = (\lambda t + \mu) \cdot 1 + t \preceq (\lambda t + \mu)t + t = (\lambda t + \mu + 1)t \quad ✓$$

# Dilated lattice polytopes

# Dilation operator

For $f \in \mathbb{R}[[t]]$ and an integer $r \geq 1$ there are uniquely determined $f_0, \ldots, f_{r-1} \in \mathbb{R}[[t]]$ such that

$$f(t) \;=\; f_0(t^r) + tf_1(t^r) + \cdots + t^{r-1} f_{r-1}(t^r).$$

For $0 \leq i \leq r-1$ we define

$$f^{\langle r,i \rangle} = f_i.$$

**Example:** $r = 2$

$$1 + 3t + 5t^2 + 7t^3 + t^5$$

Then

$$f_0 = 1 + 5t \qquad f_1 = 3 + 7t + t^2$$

In particular, for all lattice polytopes $P$ and all integers $r \geq 1$

$$\sum_{n \geq 0} \mathrm{E}_{rP}(n) t^n \;=\; \left( \sum_{n \geq 0} \mathrm{E}_P(n) t^n \right)^{\langle r,0 \rangle}$$

# $h^*$-polynomials of dilated polytopes

## Lemma (Beck, Stapledon '10)

*Let $P$ be a $d$-dimensional lattice polytope and $r \geq 1$. Then*

$$h^*_{rP}(t) = \left(h^*_P(t)(1 + t + \cdots + t^{r-1})^{d+1} d\right)^{\langle r,0 \rangle} .$$

Equivalently, for $h^*_P =: h$

$$h^*_{rP}(t) = h^{\langle r,0 \rangle} a_{d+1}^{\langle r,0 \rangle} + h^{\langle r,1 \rangle} t a_{d+1}^{\langle r,r-1 \rangle} + \cdots + h^{\langle r,r-1 \rangle} t a_{d+1}^{\langle r,1 \rangle} ,$$

where

$$a_d^{\langle r,i \rangle}(t) := \left((1 + t + \cdots + t^{r-1})^d\right)^{\langle r,i \rangle}$$

for all $r \geq 1$ and all $0 \leq i \leq r-1$.

$$
\begin{aligned}
h^*_{rP}(t) &= (1-t)^{d+1} \sum_{n\geq 0} \mathrm{E}_{rP}(n) t^n \\
&= (1-t)^{d+1} \left( \sum_{n\geq 0} \mathrm{E}_{P}(n) t^n \right)^{\langle r,0\rangle} \\
&= \left( (1-t^r)^{d+1} \sum_{n\geq 0} \mathrm{E}_{P}(n) t^n \right)^{\langle r,0\rangle} \\
&= \left( (1+t+\cdots+t^{r-1})^{d+1} (1-t)^{d+1} \sum_{n\geq 0} \mathrm{E}_{P}(n) t^n \right)^{\langle r,0\rangle} \\
&= \big( (1+t+\cdots+t^{r-1})^{d+1} h^*_P(t) \big)^{\langle r,0\rangle}
\end{aligned}
$$

# Another operator preserving interlacing...

## Proposition (Fisk '08)

*Let $f$ be a polynomial such that $f^{\langle r,r-1\rangle},\ldots,f^{\langle r,1\rangle},f^{\langle r,0\rangle}$ is an interlacing sequence. Let*

$$g(t)=(1+t+\cdots+t^{r-1})f(t)\,.$$

*Then also $g^{\langle r,r-1\rangle},\ldots,g^{\langle r,1\rangle},g^{\langle r,0\rangle}$ is an interlacing sequence.*

**Observation:**

$$\begin{pmatrix} g^{\langle r,r-1\rangle} \\ \vdots \\ g^{\langle r,1\rangle} \\ g^{\langle r,0\rangle} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ t & 1 & 1 & \cdots & 1 \\ t & t & 1 & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ t & t & \cdots & t & 1 \end{pmatrix} \begin{pmatrix} f^{\langle r,r-1\rangle} \\ \vdots \\ f^{\langle r,1\rangle} \\ f^{\langle r,0\rangle} \end{pmatrix}$$

## Corollary

*The polynomials $a_d^{\langle r,r-1\rangle}(t),\ldots,a_d^{\langle r,1\rangle}(t),a_d^{\langle r,0\rangle}(t)$ form an interlacing sequence of polynomials.*

# Putting the pieces together...

1) $h^*_{rP}(t) = h^{\langle r,0\rangle} a^{\langle r,0\rangle}_{d+1} + h^{\langle r,1\rangle} t a^{\langle r,r-1\rangle}_{d+1} + \cdots + h^{\langle r,r-1\rangle} t a^{\langle r,1\rangle}_{d+1}$

2) $a^{\langle r,r-1\rangle}_{d+1}(t), \ldots, a^{\langle r,1\rangle}_{d+1}(t), a^{\langle r,0\rangle}_{d+1}(t)$ interlacing

$\Rightarrow a^{\langle r,0\rangle}_{d+1}(t), t a^{\langle r,r-1\rangle}_{d+1}(t), \ldots, t a^{\langle r,1\rangle}_{d+1}(t)$ interlacing

**Key observation:** For $r > \deg h^*_P(t)$

$$h^{\langle r,i\rangle} = h^*_i \geq 0$$

## Theorem (J. '16)

*Let $P$ be a $d$-dimensional lattice polytope. Then $h^*_{rP}(t)$ has only real roots whenever $r \geq \deg h^*_P(t)$.*

# Stapledon Decomposition

# IDP polytopes with interior lattice points

## Question (Schepers, Van Langenhoven '13)

*For any IDP polytope $P$ with interior lattice point, is the $h^*$-polynomial $h^*_P = \sum_{i=0}^{d} h^*_i t^i$* **alternatingly increasing**, *i.e.*

$$h^*_0 \le h^*_d \le h^*_1 \le h^*_{d-1} \le \cdots \qquad ?$$

## Observation

alternatingly increasing $\Rightarrow$ unimodal with peak in the middle

- reflexive polytopes with regular unimodular triangulation✓
- lattice parallelepipeds (Schepers, Van Langenhoven '13)
- coloop-free lattice zonotopes (Beck, J., McCullough '16)

# IDP polytopes with interior lattice points

### Question

*Is there a uniform bound $N$ such that the $h^*$-polynomial of $rP$ is alternatingly increasing for all $r \geq N$?*

**Codegree**

For any $d$-dimensional lattice polytope $P$ with $\deg h^*_P = s$

$$l \ := \ \min\{r \geq 1 \colon rP^\circ \cap \mathbb{Z} \neq \emptyset\} \ = \ d + 1 - s$$

### Theorem (Higashitani '14)

*The $h^*$-polynomial of $rP$ is alternatingly increasing whenever* $r \geq \max\{s, d + 1 - s\}$.

# Stapledon Decomposition

## Theorem (Stapledon '09)

*Let $P$ be a lattice polytope with* $\deg h_P^* = s$ *and codegree* $l = d + 1 - s$. *Then* $(1 + t + \cdots + t^{l-1})h_P^*(t)$ *can be uniquely decomposed as*

$$(1 + t + \cdots + t^{l-1})h_P^*(t) \; = \; a(t) + t^l b(t)\,,$$

*where* $a(t) = t^d a(\frac{1}{t})$ *and* $b(t) = t^{d-l} b(\frac{1}{t})$ *are palindromic polynomials with nonnegative coefficients.*

**Consequences:**

$$a_i \geq 0 \Leftrightarrow h_0 + h_1 + \cdots + h_i \geq h_d + h_{d-1} + \cdots + h_{d-i+1} \quad \text{(Hibi '90)}$$

$$b_i \geq 0 \Leftrightarrow h_s + h_{s-1} + \cdots + h_i \geq h_0 + h_1 + \cdots + h_i \quad \text{(Stanley '91)}$$

# Stapledon Decomposition

## Observation

Every polynomial $h(t)$ of degree $d$ can be uniquely decomposed into palindromic polynomials $a(t) = t^d a(\frac{1}{t})$ and $b(t) = t^{d-1} b(\frac{1}{t})$ such that

$$h(t) \;=\; a(t) + t b(t)\,.$$

"Proof":

$$\begin{array}{ccccccc} & a_0 & a_1 & a_2 & a_2 & a_1 & a_0 \\ + & & b_0 & b_1 & b_2 & b_1 & b_0 \\ \hline & h_0 & h_1 & h_2 & h_3 & h_4 & h_5 \end{array}$$

## Observation

$h(t)$ is alternatingly increasing $\Leftrightarrow$ $a(t)$ and $b(t)$ are unimodal

# Stapledon Decomposition for dilated polytopes

## Theorem (J. '18+)

*Let $P$ be a lattice polytope and for all $r \geq 1$ let*

$$h^*_{rP}(t) = a_r(t) + tb_r(t)$$

*be the unique decomposition into palindromic polynomials $a_r(t) = t^d a_r(\frac{1}{t})$ and $b_r(t) = t^{d-1} b_r(\frac{1}{t})$. Then*

$$b_r(t) \preceq a_r(t)$$

*for all $r \geq d+1$.*

# Concluding remarks

- Bound for real-rootedness of $h^*_{rP}(t)$ is optimal for $\deg h^*(P)(t) \leq \frac{d+1}{2}$ (using result by Batyrev and Hofscheier '10)
- Crucial: Coefficients of $h^*$-polynomial are nonnegative. Other applications, e.g.,
  - Combinatorial positive valuations
  - Hilbert series of Cohen-Macaulay domains

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Katharina Jochemko: *On the real-rootedness of the Veronese construction for rational formal power series, International Mathematics Research Notices (online first 2017)*.

**Thank you**