Variational methods for rate- and state-dependent friction models

Ralf Kornhuber[1], Onno Oncken[2], Elias Pipping[1], Oliver Sander[1]

[1]Mathematical Institute of the Free University, Berlin, Germany
[2]GFZ German Research Centre for Geosciences, Potsdam, Germany

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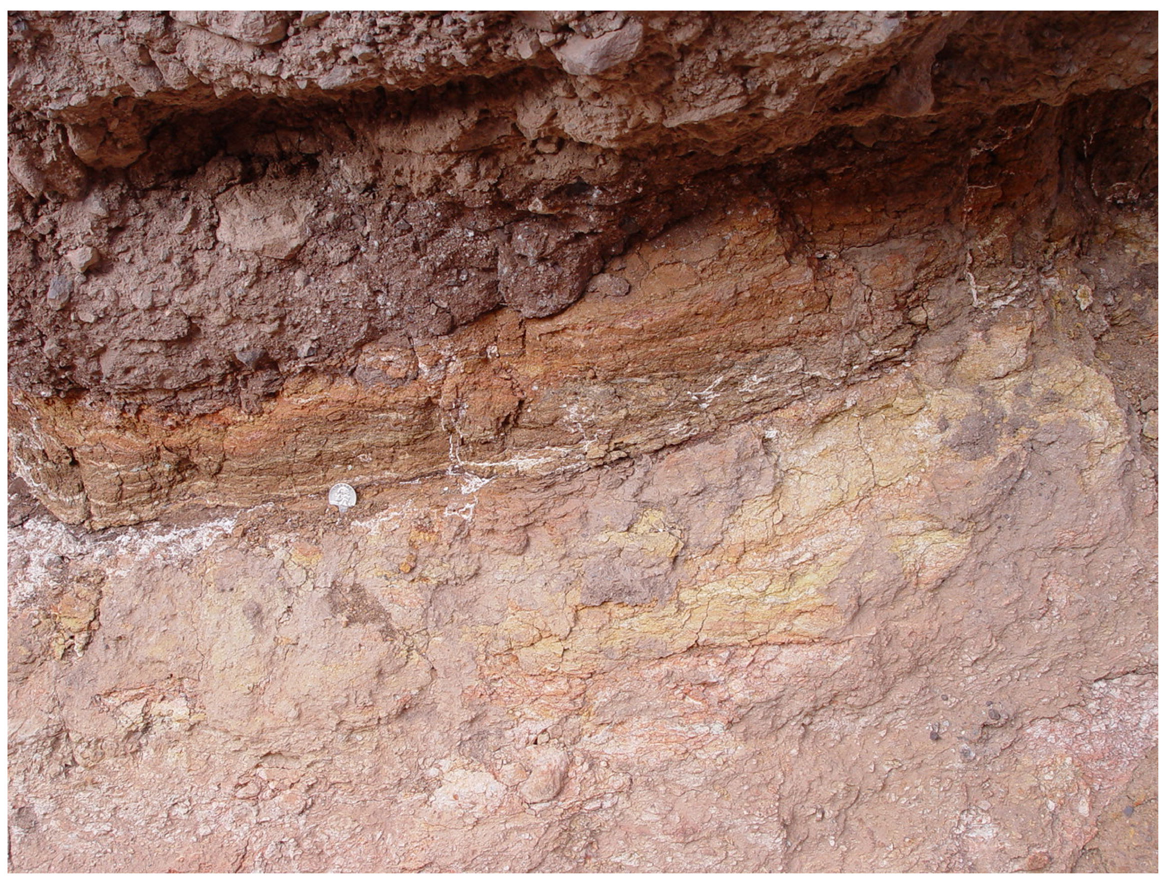
Setting



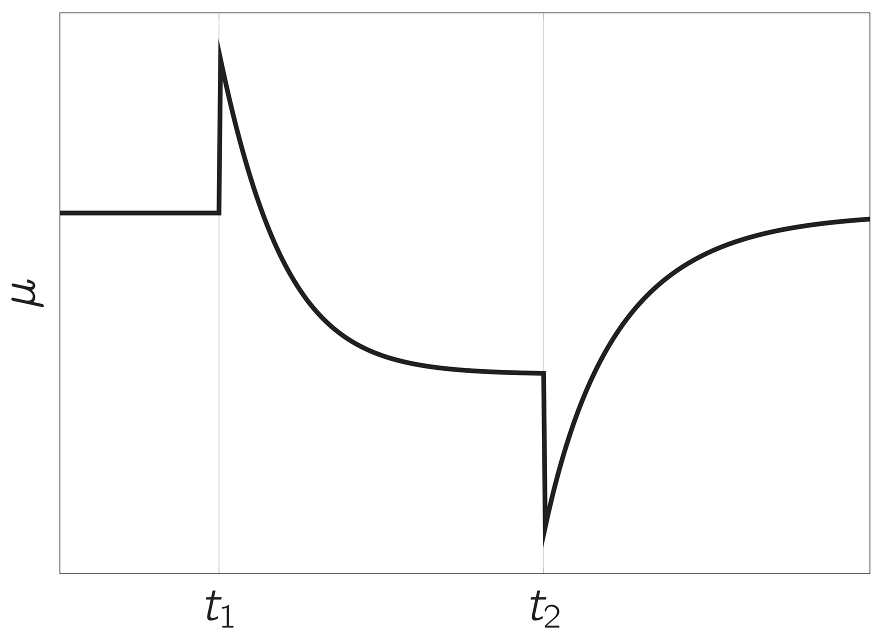
We consider two layers of rock that obey a rate- and state-dependent friction law of the form

$$\mu = \frac{\|\tau\|}{\sigma_n} = \mu_0 + a\log\frac{V}{V_0} + b\log\frac{\theta V_0}{L} \qquad (1)$$

This kind of friction law can be used to model velocity weakening, which leads to earthquakes, or velocity strengthening, which leads to fault creep.

Motivation

The above law can be motivated through velocity stepping tests, in which changes in the coefficient of friction are found to stem not only directly from a change in velocity, but also from state-effect that acts over time as shown below.



The friction coefficient under (simulated) velocity stepping

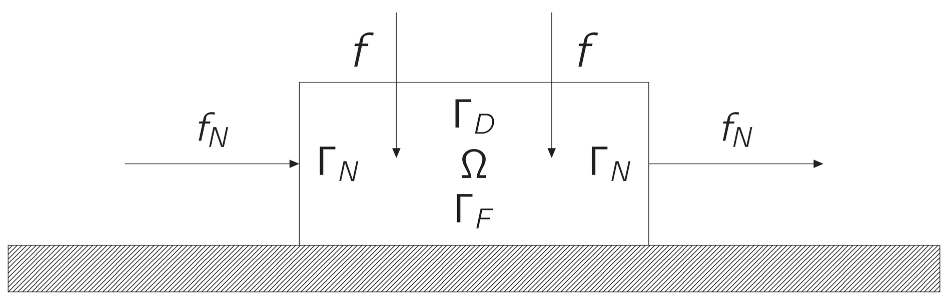
Here, the velocity in the interval $[t_1, t_2]$ is greater than the one from before t_1 and after t_2.

Model problem

We consider an elastic body Ω on a rigid surface, whose boundary is made up of three disjoint subsets

- Γ_D, on which we impose the displacement,
- Γ_N, on which we prescribe the surface force f_N, and
- Γ_F, for which we formulate the rate- and state-dependent friction law.

This situation is illustrated below.



A model slider

Assuming that acceleration can be neglected, it can be summarised as

$$\begin{aligned}
\boldsymbol{\sigma}(u) &= \mathcal{C} : \varepsilon(u) && \text{in } \Omega && \text{(elasticity)}\\
\operatorname{Div}\boldsymbol{\sigma}(u) + f &= 0 && \text{in } \Omega && \text{(balance of momentum)}\\
u &= 0 && \text{on } \Gamma_D &&\\
\sigma(u) &= f_N(t) && \text{on } \Gamma_N &&\\
u_n &= 0 && \text{on } \Gamma_F &&\\
-\tau &\in \partial_V \phi(\dot u, \theta) && \text{on } \Gamma_F && \text{(friction law)} \qquad (2)
\end{aligned}$$

We also assume that σ_n is known and bounded on Γ_F and that the state θ evolves according to either

$$\dot\theta = 1 - \frac{V}{L}\theta \quad \text{or} \quad \dot\theta = -\frac{V}{L}\theta\log\left(\frac{V}{L}\theta\right)$$

Stress function

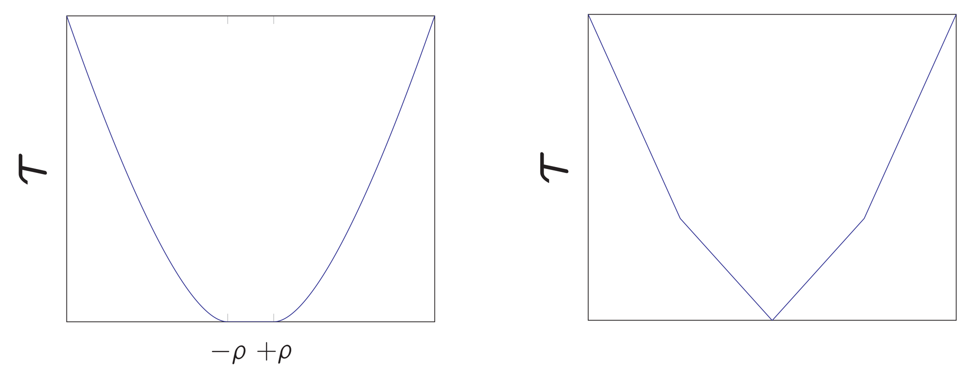
To obtain (2), we rewrite (1) as

$$\|\tau\| = a\sigma_n \log(V/V_m(\theta)) = a\sigma_n \frac{\partial F}{\partial V}(V,\theta)$$

with $V_m(\theta) = V\exp(-\mu/a)$ and

$$F(V,\theta) := V\log(V/V_m(\theta)) - V + V_m(\theta)$$

With state fixed, the tangential stress is thus the derivative of a function wrt. velocity. That function is plotted below (on the left).



The function $F(\cdot,\theta)$ A sample function G

Tangential stress over velocity

While F is very smooth, we could just as easily consider a non-smooth convex function like G. In particular, we can handle the unregularised friction law.

Time-discrete problem

Through time discretisation we arrive at a system of two coupled elliptic variational inequalities of the form

$$\exists u\colon\ a(u, v-u) + j(\theta, v) - j(\theta, u) \geq \ell(v-u)\ \forall v$$

and

$$\exists \theta\colon\ A(\theta, \vartheta-\theta) + J(u,\vartheta) - J(u,\theta) \geq L(\vartheta-\theta)\ \forall\vartheta$$

making our problem accessible to modern analytical tools as well as fast and robust numerical algorithms.

As a consequence, both problems are found to possess energies whose minima are attained solely at the respective solutions. The coupled minimisation problems that result from this observation are solved using a fixed-point iteration.

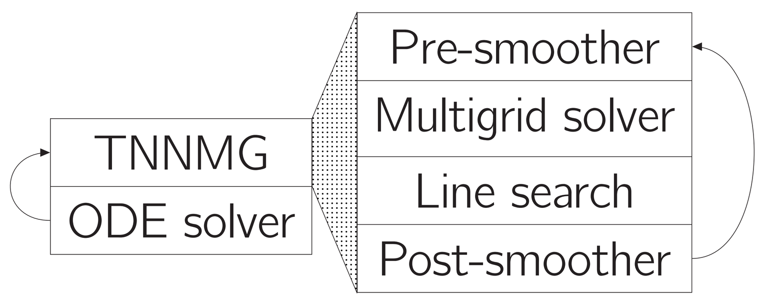
Numerical framework of choice

Dune

Distributed and Unified Numerics Environment

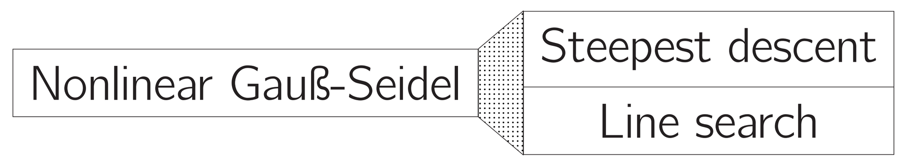
Algorithm

Once we have discretised in space, Finite Elements are used. The displacement problem is solved using TNNMG, the Truncated Nonsmooth Newton Multigrid method. The state problem decouples pointwise, leading to scalar minimisation problems.



The overall algorithm

The TNNMG method involves smoothing that guarantees convergence and a multigrid solver, to which it owes its speed.

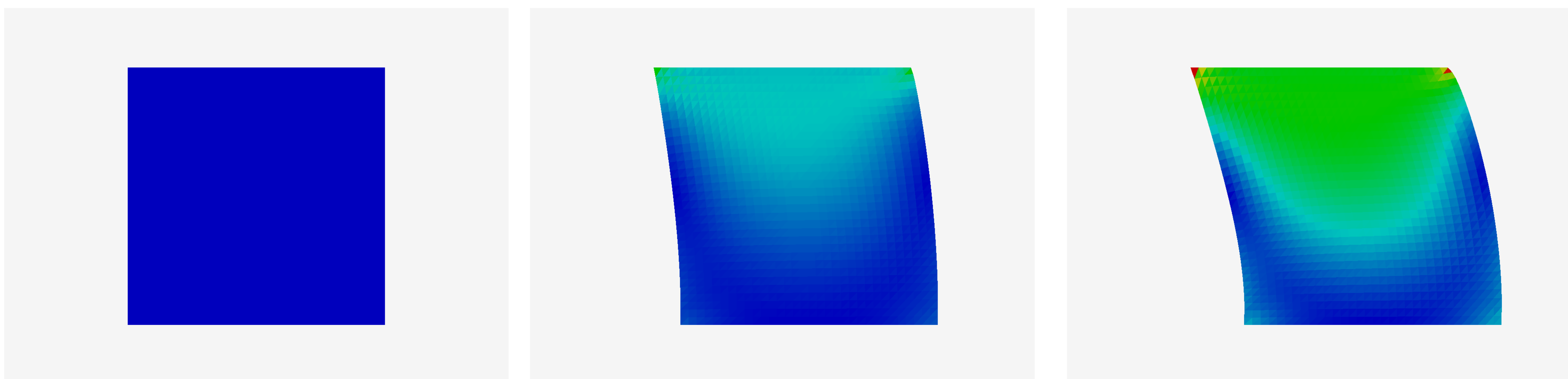


Detail: The smoother

At each node, the nonlinear Gauß-Seidel smoother solves minimisation problems using steepest descent.

Deforming slider

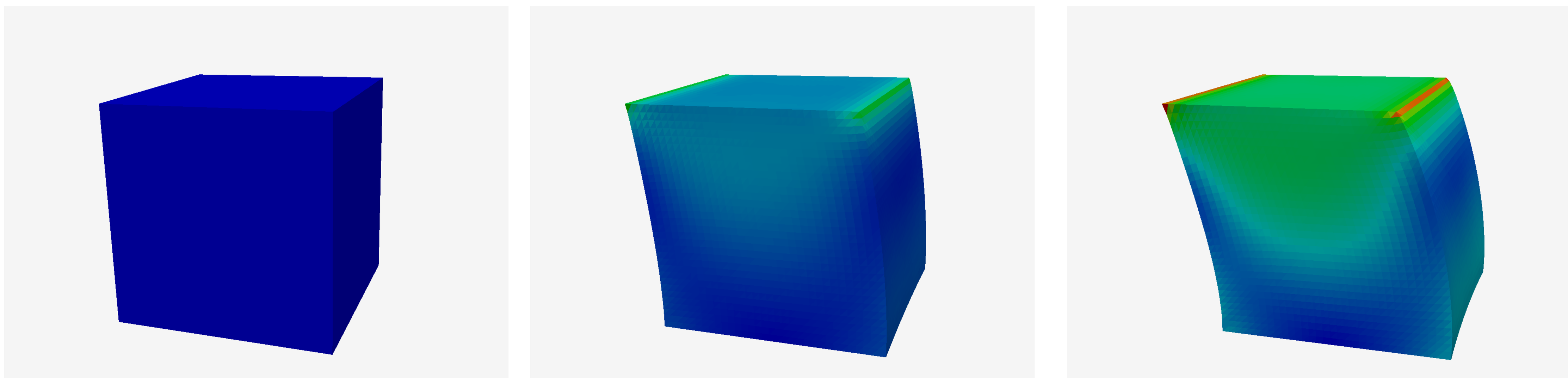
In this two-dimensional instance of the model problem, the body is pressed to the right while its top is fixed and its bottom is allowed to slide. A numerical simulation can be seen below.



t = 0 t = 250 t = 500

A sliding example in 2D (with 1600x magnified deformation)

The same code can handle the three-dimensional case.

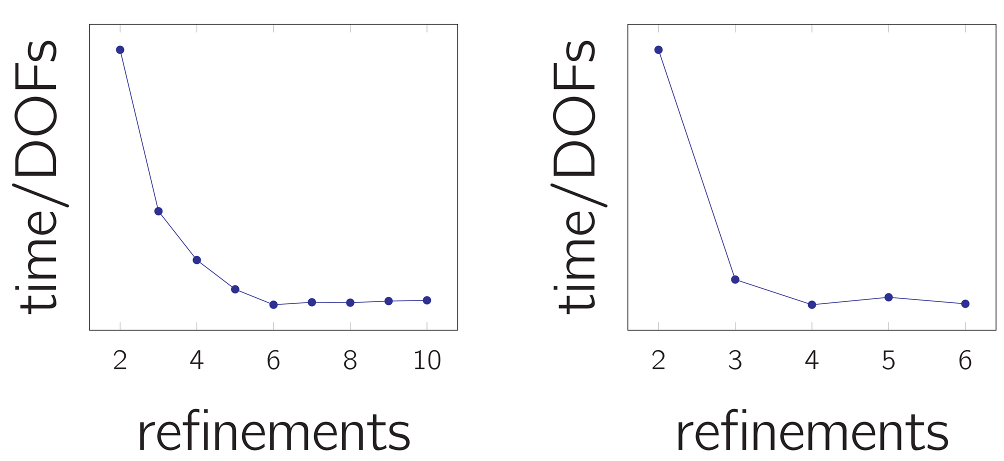


t = 0 t = 250 t = 500

A sliding example in 3D (with 1600x magnified deformation)

Computational effort and degrees of freedom

The computational effort (measured in wall clock time) is eventually linear in the degrees of freedom as can be seen in the graph below (l: 2D, r: 3D). Our algorithm is thus optimal.



Solving a 3D problem with 262144 elements and 500 timesteps up to an accuracy of 10^{-14} takes 8.5 hours on a single core of a current processor (Intel Xeon E31245, 3.30GHz). The table below gives precise timings.

ref.	2	3	4	5	6	7	8	9	10
2D	1	3	9	30	110	446	1779	7197	28945
3D	13	65	489	3990	31378				

Time in seconds for various levels of refinement

Mail: kornhuber@math.fu-berlin.de, onno.oncken@gfz-potsdam.de, pipping@math.fu-berlin.de, sander@math.fu-berlin.de