Subduction Zone Simulations with Rate-and-State Friction

E. Pipping[1], R. Kornhuber[1], M. Rosenau[2], O. Oncken[2]

[1]Mathematisches Institut, Freie Universität Berlin,
[2]Geologische Systeme: Lithosphärendynamik, GeoForschungsZentrum Potsdam

1st of July 2015

Overview / motivation

Setting: Single, pre-existing fault. Small deformations.
No pore fluids, no temperature dependence.

Overview / motivation

Setting: Single, pre-existing fault. Small deformations.
No pore fluids, no temperature dependence.

Aim: Simulate complete seismic cycles with rate-and-state friction.
Fully dynamic, no quasistatic or quasidynamic approximation.

Overview / motivation

Setting: Single, pre-existing fault. Small deformations.
No pore fluids, no temperature dependence.

Aim: Simulate complete seismic cycles with rate-and-state friction.
Fully dynamic, no quasistatic or quasidynamic approximation.

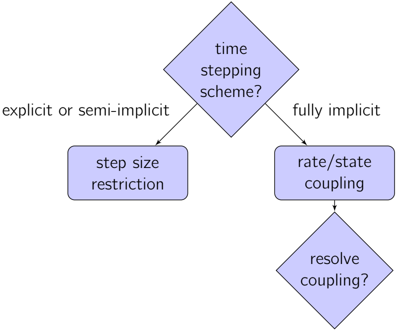
Approach: Rothe's method (spatial discretisation: FEM)

Overview / motivation

Setting: Single, pre-existing fault. Small deformations.
No pore fluids, no temperature dependence.

Aim: Simulate complete seismic cycles with rate-and-state friction.
Fully dynamic, no quasistatic or quasidynamic approximation.

Approach: Rothe's method (spatial discretisation: FEM)

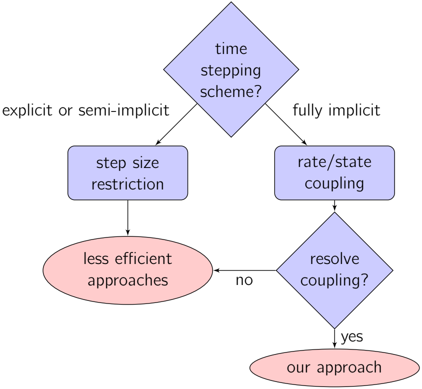


Overview / motivation

Setting: Single, pre-existing fault. Small deformations.
No pore fluids, no temperature dependence.

Aim: Simulate complete seismic cycles with rate-and-state friction.
Fully dynamic, no quasistatic or quasidynamic approximation.

Approach: Rothe's method (spatial discretisation: FEM)



Rate-and-state friction

Rate-and-state friction laws by Dieterich/Ruina (1983),

$$\mu(V,\alpha) = \mu_* + a \log \frac{V}{V_*} + b \underbrace{\log \frac{\theta V_*}{L}}_{\alpha}, \quad \dot{\theta}(\theta, V) = \begin{cases} 1 - \frac{\theta V}{L} & \text{ageing law} \\ -\frac{\theta V}{L} \log \frac{\theta V}{L} & \text{slip law} \end{cases}$$

Rate-and-state friction

Rate-and-state friction laws by Dieterich/Ruina (1983),

$$\mu(V,\alpha) = \mu_* + a\log\frac{V}{V_*} + b\underbrace{\log\frac{\theta V_*}{L}}_{\alpha}, \quad \dot{\theta}(\theta, V) = \begin{cases} 1 - \frac{\theta V}{L} & \text{ageing law} \\ -\frac{\theta V}{L}\log\frac{\theta V}{L} & \text{slip law} \end{cases}$$

$$= a\log\left(\frac{V}{V_*}\exp\left(\frac{\mu_* + b\alpha}{a}\right)\right)$$

Rate-and-state friction

Rate-and-state friction laws by Dieterich/Ruina (1983),

$$\mu(V,\alpha) = \mu_* + a\log\frac{V}{V_*} + b\underbrace{\log\frac{\theta V_*}{L}}_{\alpha}, \quad \dot{\theta}(\theta, V) = \begin{cases} 1 - \frac{\theta V}{L} & \text{ageing law} \\ -\frac{\theta V}{L}\log\frac{\theta V}{L} & \text{slip law} \end{cases}$$

$$= a\log\left(\frac{V}{V_*}\exp\left(\frac{\mu_* + b\alpha}{a}\right)\right)$$

Regularisation by Rice/Ben-Zion (1996)

$$\approx a\sinh^{-1}\left(\frac{V}{2V_*}\exp\left(\frac{\mu_* + b\alpha}{a}\right)\right) \geq 0$$

Rate-and-state friction

Rate-and-state friction laws by Dieterich/Ruina (1983),

$$\mu(V,\alpha) = \mu_* + a\log\frac{V}{V_*} + b\underbrace{\log\frac{\theta V_*}{L}}_{\alpha}, \quad \dot{\theta}(\theta, V) = \begin{cases} 1 - \frac{\theta V}{L} & \text{ageing law} \\ -\frac{\theta V}{L}\log\frac{\theta V}{L} & \text{slip law} \end{cases}$$

$$= a\log\left(\frac{V}{V_*}\exp\left(\frac{\mu_* + b\alpha}{a}\right)\right)$$

Regularisation by Rice/Ben-Zion (1996)

$$\approx a\sinh^{-1}\left(\frac{V}{2V_*}\exp\left(\frac{\mu_* + b\alpha}{a}\right)\right) \geq 0$$

Another regularisation

$$\approx \mu_* + a\log\left(\frac{V}{V_*} + 1\right) + b\alpha$$

Rate-and-state friction

Rate-and-state friction laws by Dieterich/Ruina (1983),

$$\mu(V,\alpha) = \mu_* + a\log\frac{V}{V_*} + b\underbrace{\log\frac{\theta V_*}{L}}_{\alpha}, \quad \dot\theta(\theta, V) = \begin{cases} 1 - \frac{\theta V}{L} & \text{ageing law} \\ -\frac{\theta V}{L}\log\frac{\theta V}{L} & \text{slip law} \end{cases}$$

$$= a\log\left(\frac{V}{V_*}\exp\left(\frac{\mu_* + b\alpha}{a}\right)\right)$$

Regularisation by Rice/Ben-Zion (1996)

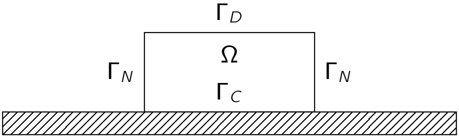
$$\approx a\sinh^{-1}\left(\frac{V}{2V_*}\exp\left(\frac{\mu_* + b\alpha}{a}\right)\right) \geq 0$$

Another regularisation

$$\approx \mu_* + a\log\left(\frac{V}{V_*} + 1\right) + b\alpha$$

Common assumption: Constant normal stress

Prototypical one-body problem



$$
\begin{aligned}
\boldsymbol{\sigma}(\mathbf{u}) &= \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) + \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) && \text{in } \Omega && \text{(linear viscoelasticity)} \\
\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{b} &= \rho \ddot{\mathbf{u}} && \text{in } \Omega && \text{(momentum balance)} \\
\dot{\mathbf{u}}_n &= 0 && \text{on } \Gamma_C && \text{(bilateral contact)} \\
\boldsymbol{\sigma}_t = -\lambda \dot{\mathbf{u}}, \quad |\boldsymbol{\sigma}_t| = \lambda |\dot{\mathbf{u}}| &= |\sigma_n| \mu(|\dot{\mathbf{u}}|, \alpha) + C && \text{on } \Gamma_C && \text{with } \lambda = 0 \text{ for } \dot{\mathbf{u}} = 0 \\
&\dots && \text{on } \Gamma_{N,D} && \\
\dot{\alpha} &= \dot{\alpha}(|\dot{\mathbf{u}}|, \alpha) && \text{on } \Gamma_C && \text{(family of ODEs)}
\end{aligned}
$$

Towards spatial discretisation: Weak formulation

We get

$$\int_\Omega \rho\ddot{\mathbf{u}}(\mathbf{v}-\dot{\mathbf{u}}) + \int_\Omega \mathcal{B}\boldsymbol{\varepsilon}(\dot{\mathbf{u}})\colon \boldsymbol{\varepsilon}(\mathbf{v}-\dot{\mathbf{u}}) + \int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u})\colon \boldsymbol{\varepsilon}(\mathbf{v}-\dot{\mathbf{u}}) + \int_{\Gamma_C} \phi(\mathbf{v},\alpha) \geq \int_{\Gamma_C} \phi(\dot{\mathbf{u}},\alpha) + \ell(\mathbf{v}-\dot{\mathbf{u}})$$

for every $\mathbf{v} \in \mathcal{H}$ with

$$\mathcal{H} = \{\mathbf{v} \in H^1(\Omega)^d \colon \mathbf{v} = 0 \text{ on } \Gamma_D,\ \mathbf{v}_n = 0 \text{ on } \Gamma_C\}$$

or briefly

$$0 \in M\ddot{\mathbf{u}} + C\dot{\mathbf{u}} + A\mathbf{u} + \partial\Phi(\,\cdot\,,\alpha)(\dot{\mathbf{u}}) - \ell \subset \mathcal{H}^*$$

and

$$\dot{\alpha} = \dot{\alpha}(|\dot{\mathbf{u}}|,\alpha) \quad \text{a.e. on } \Gamma_C$$

Implicit time discretisation

Starting point:

$$
\begin{aligned}
0 &\in M\ddot{\mathbf{u}} + C\dot{\mathbf{u}} + A\mathbf{u} + \partial\Phi(\,\cdot\,,\alpha)(\dot{\mathbf{u}}) - \ell \\
\dot{\alpha} &= \dot{\alpha}(|\dot{\mathbf{u}}|,\alpha)
\end{aligned}
\qquad \text{(S)}
$$

Implicit time discretisation

Starting point:

$$0 \in M\ddot{\mathbf{u}} + C\dot{\mathbf{u}} + A\mathbf{u} + \partial\Phi(\,\cdot\,,\alpha)(\dot{\mathbf{u}}) - \ell$$
$$\dot{\alpha} = \dot{\alpha}(|\dot{\mathbf{u}}|,\alpha)$$

After collocation (rate), approximation of $\dot{\mathbf{u}}$ on $[t_{n-1}, t_n]$ (state):

$$0 \in M\ddot{\mathbf{u}}_n + C\dot{\mathbf{u}}_n + A\mathbf{u}_n + \partial\Phi(\,\cdot\,,\alpha_n)(\dot{\mathbf{u}}_n) - \ell_n$$
$$\dot{\alpha} = \dot{\alpha}(|\dot{\mathbf{u}}_{n-\lambda}|,\alpha) \quad \text{with } \dot{\mathbf{u}}_{n-\lambda} = \lambda\dot{\mathbf{u}}_{n-1} + (1-\lambda)\dot{\mathbf{u}}_n \quad (0 \le \lambda < 1)$$

(S)

Implicit time discretisation

Starting point:

$$0 \in M\ddot{\mathbf{u}} + C\dot{\mathbf{u}} + A\mathbf{u} + \partial\Phi(\,\cdot\,,\alpha)(\dot{\mathbf{u}}) - \ell$$
$$\dot{\alpha} = \dot{\alpha}(|\dot{\mathbf{u}}|,\alpha)$$

After collocation (rate), approximation of $\dot{\mathbf{u}}$ on $[t_{n-1}, t_n]$ (state):

$$0 \in M\ddot{\mathbf{u}}_n + C\dot{\mathbf{u}}_n + A\mathbf{u}_n + \partial\Phi(\,\cdot\,,\alpha_n)(\dot{\mathbf{u}}_n) - \ell_n$$
$$\dot{\alpha} = \dot{\alpha}(|\dot{\mathbf{u}}_{n-\lambda}|,\alpha) \quad \text{with } \dot{\mathbf{u}}_{n-\lambda} = \lambda\dot{\mathbf{u}}_{n-1} + (1-\lambda)\dot{\mathbf{u}}_n \quad (0 \leq \lambda < 1)$$

After time discretisation (rate), determining the flow operator (state)

$$0 \in \left(\frac{\lambda_M}{\tau}M + C + \frac{\tau}{\lambda_A}A\right)\dot{\mathbf{u}}_n + \partial\Phi(\,\cdot\,,\alpha_n)(\dot{\mathbf{u}}_n) - \ell_n - \dots \qquad \text{(R)}$$
$$\alpha_n = \Psi^\tau(|\dot{\mathbf{u}}_{n-\lambda}|,\alpha_{n-1}) \qquad \text{(S)}$$

Implicit time discretisation

Starting point:

$$0 \in M\ddot{\mathbf{u}} + C\dot{\mathbf{u}} + A\mathbf{u} + \partial\Phi(\,\cdot\,,\alpha)(\dot{\mathbf{u}}) - \ell$$
$$\dot{\alpha} = \dot{\alpha}(|\dot{\mathbf{u}}|,\alpha)$$

After collocation (rate), approximation of $\dot{\mathbf{u}}$ on $[t_{n-1}, t_n]$ (state):

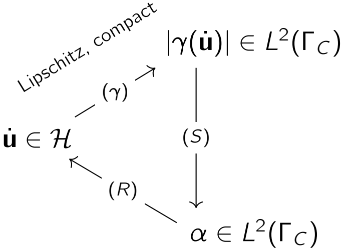
$$0 \in M\ddot{\mathbf{u}}_n + C\dot{\mathbf{u}}_n + A\mathbf{u}_n + \partial\Phi(\,\cdot\,,\alpha_n)(\dot{\mathbf{u}}_n) - \ell_n$$
$$\dot{\alpha} = \dot{\alpha}(|\dot{\mathbf{u}}_{n-\lambda}|,\alpha) \quad \text{with } \dot{\mathbf{u}}_{n-\lambda} = \lambda\dot{\mathbf{u}}_{n-1} + (1-\lambda)\dot{\mathbf{u}}_n \quad (0 \le \lambda < 1)$$

After time discretisation (rate), determining the flow operator (state)

$$0 \in \left(\frac{\lambda_M}{\tau}M + C + \frac{\tau}{\lambda_A}A\right)\dot{\mathbf{u}}_n + \partial\Phi(\,\cdot\,,\alpha_n)(\dot{\mathbf{u}}_n) - \ell_n - \dots \tag{R}$$
$$\alpha_n = \Psi^\tau(|\dot{\mathbf{u}}_{n-\lambda}|,\alpha_{n-1}) \tag{S}$$

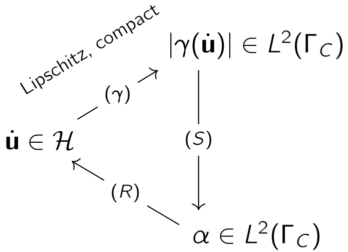
Structure: (R) Positive rate effect $\rightsquigarrow$ convex minimisation!
(S) Trivial.

Rate/state coupling



$$T\colon \mathcal{H} \to \mathcal{H} \begin{cases} (S) \text{ solve ODEs} \\ (R) \text{ convex minimisation} \\ (\gamma) \text{ trace map + norm} \end{cases}$$

Rate/state coupling



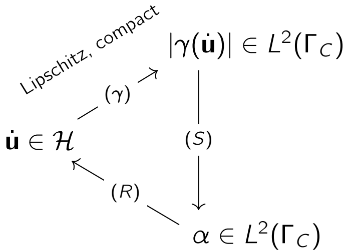
$$T\colon \mathcal{H} \to \mathcal{H} \begin{cases} (S) \text{ solve ODEs} \\ (R) \text{ convex minimisation} \\ (\gamma) \text{ trace map + norm} \end{cases}$$

Analytic findings: Contraction if

- Ageing law
- Non-zero Viscosity
- τ small enough

We then have: Existence, uniqueness, convergence ($\rightsquigarrow$ algorithm!).

Rate/state coupling



$$T : \mathcal{H} \to \mathcal{H} \begin{cases} (S) \text{ solve ODEs} \\ (R) \text{ convex minimisation} \\ (\gamma) \text{ trace map + norm} \end{cases}$$

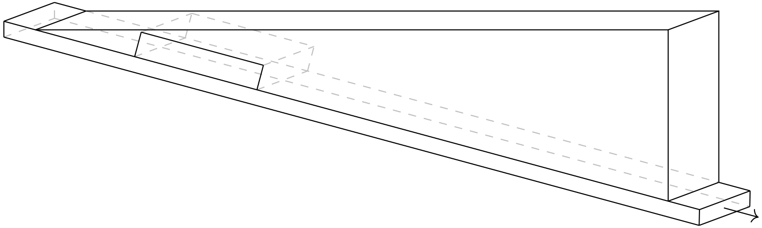
Analytic findings: Contraction if

- Ageing law
- Non-zero Viscosity
- τ small enough

We then have: Existence, uniqueness, convergence ($\rightsquigarrow$ algorithm!).

More generally: Existence.

Application: a simplified subduction zone



Lower plate moves at a prescribed velocity, right end held fixed.

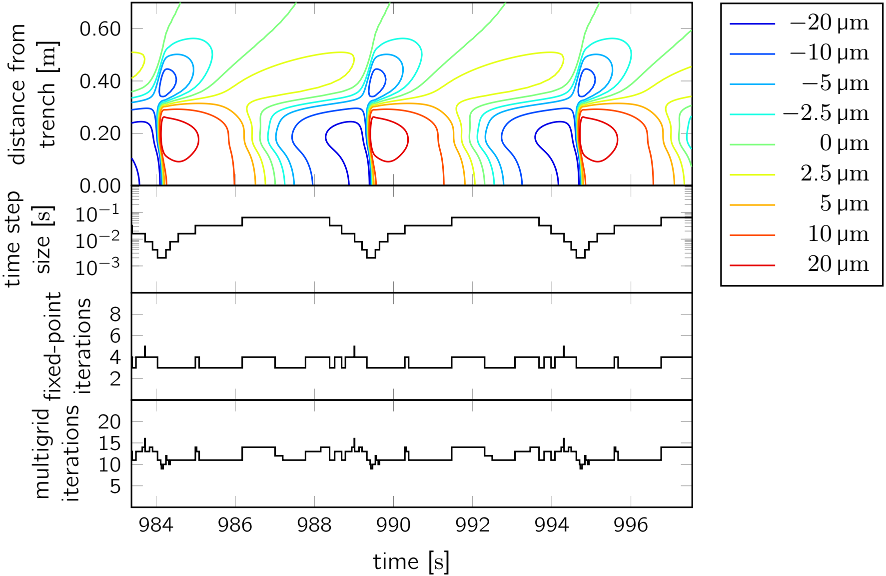


Figure: Vertical surface displacement (relative to a time average).

Comparison: simulation and experiment

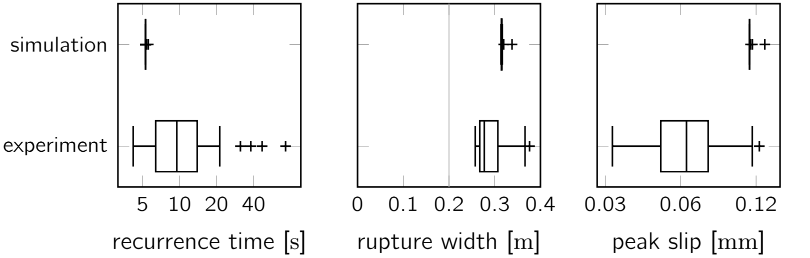


Figure: Tukey boxplots for recurrence time, rupture width, and peak slip.

Is complete resolution of the coupling efficient?

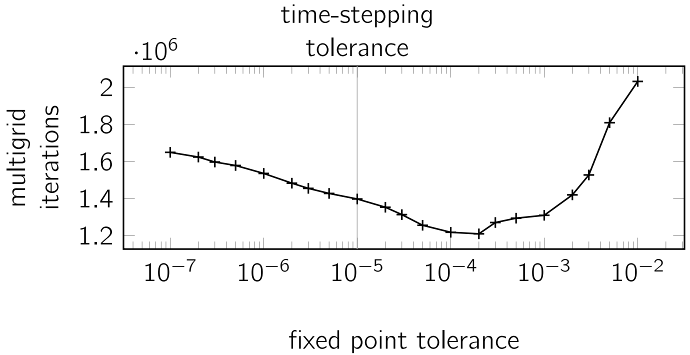
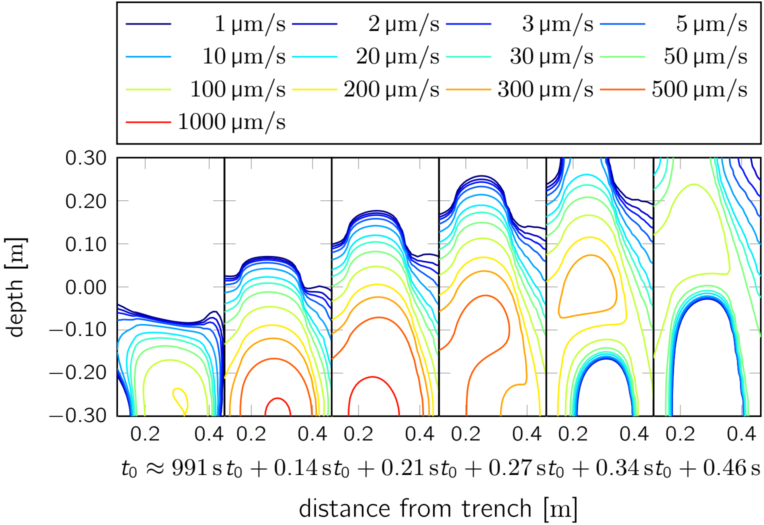


Figure: Computational effort over the prescribed error tolerance ε.

Snapshots from a 3D simulation



Conclusion / outlook

Status quo

- Robust, efficient solver.
  Corner stone: nonlinear multigrid "TNNMG".
- Open source implementation in C++.
  Foundation: DUNE framework

Outlook

- 2-body problems
- Plasticity
- Parallelisation
- Integration into multiscale fault network

Further reading

- E. Pipping, O. Sander and R. Kornhuber. "Variational formulation of rate- and state-dependent friction problems". In: *Zeitschrift für Angewandte Mathematik und Mechanik. Journal of Applied Mathematics and Mechanics* (2013). ISSN: 1521-4001. DOI: 10.1002/zamm.201300062.
- E. Pipping. "Dynamic problems of rate-and-state friction in viscoelasticity". Dissertation. Freie Universität Berlin, 2014. URN: urn:nbn:de:kobv:188-fudissthesis000000098145-4.
- E. Pipping, R. Kornhuber, M. Rosenau and O. Oncken. "Numerical approximation of rate-and-state friction problems". 2015. URL: http://publications.mi.fu-berlin.de/1538/. Forthcoming.